p-adic properties of sequences and finite state automata

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Arian Daneshvar, Pujan Dave, Zhefan Wang
Amita Malik (team leader), Armin Straub (faculty mentor)

Illinois Geometry Lab

Motivation: Fibonacci numbers

- $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$
- **Recursive relation:**

$$F(-1) = 0, \quad F(0) = 1,$$
$$F(n+1) = F(n) + F(n-1) \text{ for } n \geq 0.$$

- **Generating function:**
$F(n)$ are given by the *Taylor coefficients* of
$$\frac{1}{1 - x - x^2}.$$

Apéry Numbers

- $1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$
- **Recursive relation:** $(a, b, c, d) = (17, 5, 1, 0)$

$$A(-1) = 0, \quad A(0) = 1,$$
$$A(n+1) = \frac{(2n+1)(an^2 + an + b)A(n) - n(cn^2 + d)A(n-1)}{(n+1)^3}.$$

- **Binomial sum:**
$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- **Generating function:**
Apéry numbers $A(n)$ are given by the *diagonal Taylor coefficients* of
$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

- **History:**
Used by Roger Apéry in 1978 to prove that
$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$
is irrational. Still not known whether $\zeta(5)$ is irrational!

Apéry-like sequences

- Conjecturally, the above recurrence has very few (up to scaling) integer sequences as solutions:
  - polynomial solutions (like $B(n) = n^2 + (n+1)^2$),
  - terminating solutions (like $1, 1, 0, 0, 0, \ldots$),
  - hypergeometric solutions (like $B(n) = \binom{2n}{n}^3$),
  - related Legendrian solutions,
  - 6+6+3 sporadic solutions, including:

$$D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$$
Domb numbers (10,4,64,0)

$$C(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}$$
one of Cooper's sequences (13,4,-27,3)

In the p-adic universe

- The Apéry numbers $A(n)$ modulo a prime (power):

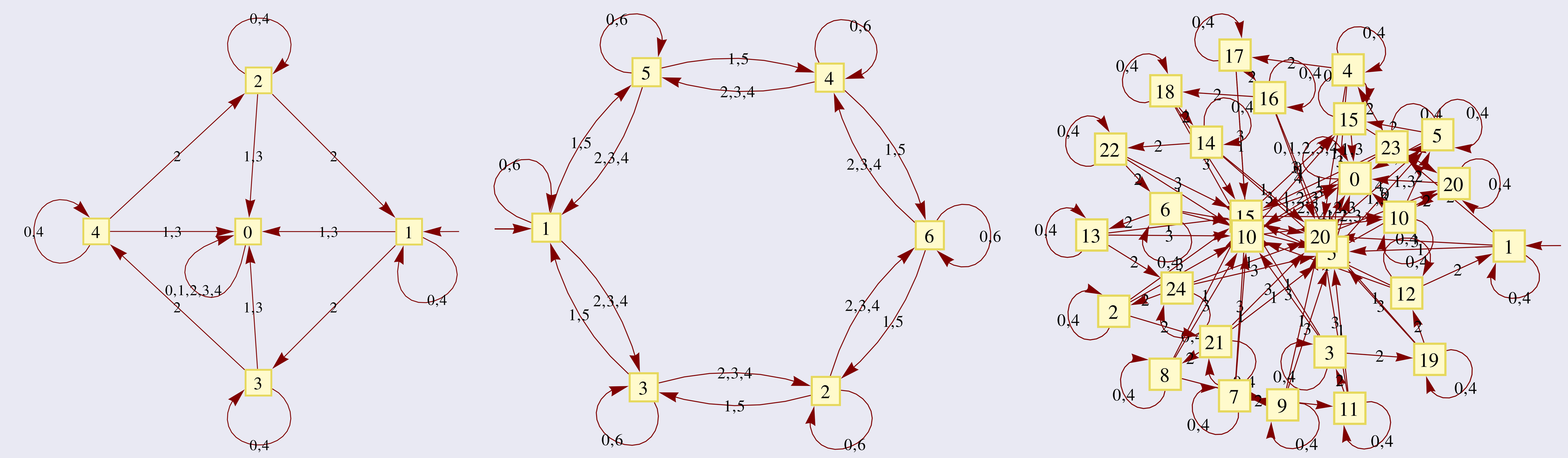
$A(n) \pmod 3$: $1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots$
$A(n) \pmod 5$: $1, 0, 3, 0, 1, 0, 0, 0, 0, 0, 3, 0, 4, 0, 3, 0, 0, 0, 0, 0, 1, 0, 3, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$
$A(n) \pmod 7$: $1, 5, 3, 3, 3, 5, 1, 5, 4, 1, 1, 1, 4, 5, 3, 1, 2, 2, 2, 1, 3, 3, 1, 2, 2, 2, 1, 3, 3, 1, 2, 2, 2, 1, 3, 5, 4, 1, 1, 1, 4, \ldots$
$A(n) \pmod{5^2}$: $1, 5, 23, 20, 1, 5, 0, 15, 0, 5, 23, 0, 4, 0, 23, 20, 0, 10, 0, 20, 1, 20, 23, 5, 1, 5, 0, 15, 0, 5, 0, 0, 0, 0, \ldots$

Observations:
- periodic modulo 3
- no Apéry number divisible by 3 or 7

Lucas congruences:
$$A(np + m) \equiv A(n)A(m) \pmod p$$
For instance, $A(514) = A(4024_{\text{base } 5}) \equiv A(4)A(0)A(2)A(4) \equiv 3 \pmod 5$.

Finite state automata

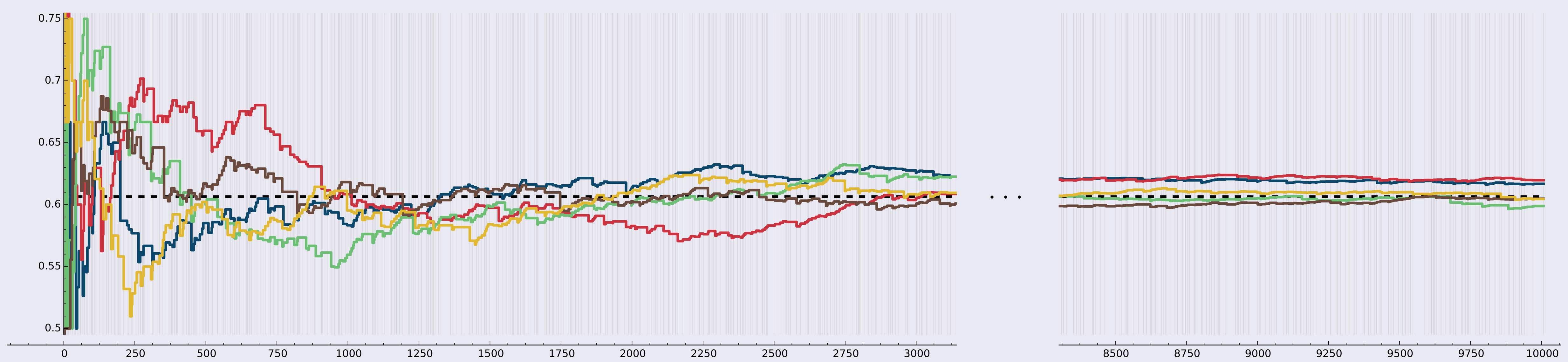
- For a sequence with (multivariate) rational generating function, work of Furstenberg, Deligne, Denef and Lipshitz implies that the values modulo p can be produced by a finite state automaton.
- The automata for the Apéry numbers modulo 5, 7, 25:

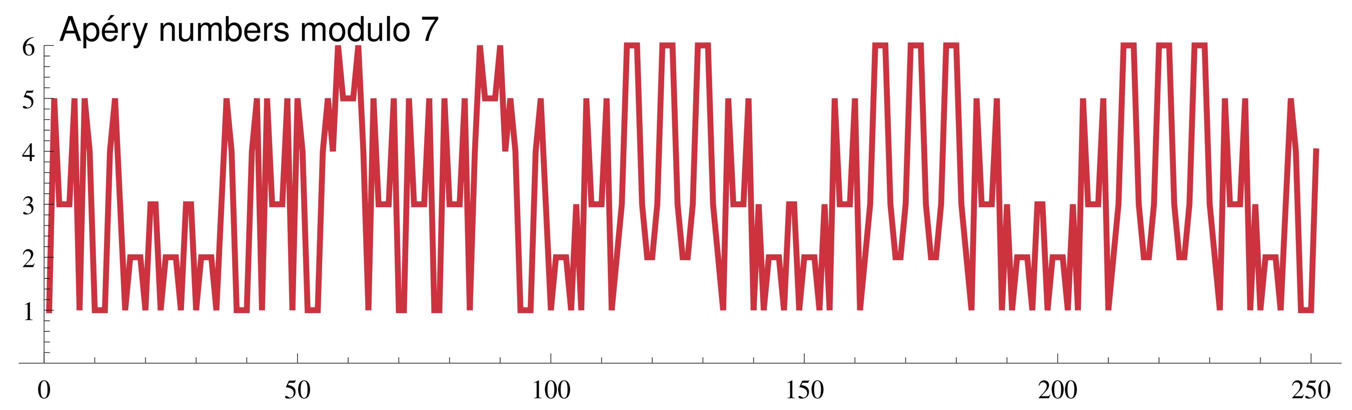


Again, for instance, $A(514) = A(4024_{\text{base } 5}) \equiv 3 \pmod 5$.

Primes not dividing a given Apéry-like sequence

- Apéry numbers (blue): $2, 3, 7, 13, 23, 29, 43, 47, 53, 67, 71, 79, 83, 89, \ldots$
- Almkvist–Zudilin numbers (red): $2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97, \ldots$
- Domb numbers (green): $3, 5, 13, 17, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 83, 89, \ldots$
- Experimentally: about 60% each (for 5 sequences). Could it be $e^{-1/2} \approx 0.607$, as a heuristic argument suggests?





Results of our IGL project

- It was shown by Chowla–Cowles–Cowles in 1980 that the Apéry numbers are periodic modulo 8. We prove that the Almkvist–Zudilin numbers
$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$
are periodic modulo 8 as well.
- For all Apéry-like sequences, we experimentally determined modulo which prime powers they are periodic (four more cases; only modulo 2 or 3).

Ongoing treasure hunt

- Lucas congruences for all Apéry-like sequences
- Prove that our experimental data on periods of Apéry-like sequences modulo prime powers is complete and accurate.
- For one sequence, namely $(11, 5, 125, 0)$, we observe that about 31% of the primes don't divide any of its terms. How about a heuristic explanation?

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