Midterm #2

MATH 332 — Differential Equations II
Friday, Apr 11, 2025

Please print your name:

No notes, fancy calculators or tools of any kind are permitted. There are 31 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (3 points) Let $y(x)$ be the unique solution to the IVP $y'' = (x+1)y^2 + 1,\ y(0) = 2,\ y'(0) = 3.$

Determine the first several terms (up to x^3) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = (0+1) \cdot y(0)^2 + 1 = 5$.

Differentiating both sides of the DE, we obtain $y''' = y^2 + 2(x+1)\, yy'$.

In particular, $y'''(0) = y(0)^2 + 2 \cdot (0+1) \cdot y(0)\, y'(0) = 16$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots = 2 + 3x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \ldots$

Problem 2. (3 points) A mass-spring system is described by the DE $my'' + 7y = F(t)$ where $F(t)$ is an external force with period 3. For which values of m can resonance occur?

Solution. $F(t)$ has a Fourier series of the form $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{3}\right) + b_n \sin\left(\frac{2\pi n t}{3}\right) \right)$.

The roots of $p(D) = mD^2 + 7$ are $\pm i\sqrt{\frac{7}{m}}$, so that the natural frequency is $\sqrt{\frac{7}{m}}$. Resonance therefore can occur if $\sqrt{\frac{7}{m}} = \frac{2\pi n}{3}$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance can occur if $m = \frac{63}{4\pi^2 n^2}$ for some $n \in \{1, 2, 3, \ldots\}$.

Problem 3. (3 points) Find a minimum value for the radius of convergence of a power series solution to

$$(x-3)y'' = \frac{2y+1}{x^2+1} \quad \text{at } x = 1.$$

Solution. Note that this is a linear DE! (Otherwise, we could not proceed.) Rewriting the DE as $y'' - \frac{2}{(x^2+1)(x-3)}y = \frac{1}{(x^2+1)(x-3)}$, we see that the singular points are $x = \pm i, 3$.

Note that $x = 1$ is an ordinary point of the DE and that the distance to the nearest singular point is $|1 - (\pm i)| = \sqrt{1^2 + 1^2} = \sqrt{2}$ (the distance to 3 is $|1-3| = 2 > \sqrt{2}$).

Hence, the DE has power series solutions about $x = 1$ with radius of convergence at least $\sqrt{2}$.

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Problem 4. (6 points) Derive a recursive description of a power series solution $y(x)$ (around $x=0$) to the differential equation $y''=3y-x^2y'$.

Solution. Let us spell out the power series for y, x^2y', y'':

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$x^2y'(x)=\sum_{n=1}^{\infty} n a_n x^{n+1}=\sum_{n=2}^{\infty} (n-1)a_{n-1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n=3\sum_{n=0}^{\infty} a_n x^n-\sum_{n=2}^{\infty} (n-1)a_{n-1}x^n.$$

We compare coefficients of x^n:

- $n=0$: $2a_2=3a_0$, so that $a_2=\frac{3}{2}a_0$.
- $n=1$: $6a_3=3a_1$, so that $a_3=\frac{1}{2}a_1$.
- $n\geqslant 2$: $(n+2)(n+1)a_{n+2}=3a_n-(n-1)a_{n-1}$

  Equivalently, for $n\geqslant 4$, $a_n=\frac{3}{n(n-1)}a_{n-2}-\frac{(n-3)}{n(n-1)}a_{n-3}$. (Can you see why this also holds for $n=3$?)

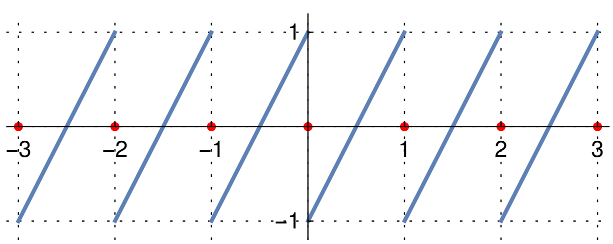
The values a_0 and a_1 are the initial conditions, and the above provides the values of a_n for $n\geqslant 2$.

Problem 5. (4 points) Consider the function $f(t)=2t-1$, defined for $t\in[0,1]$. Sketch the following for $t\in[-3,3]$.

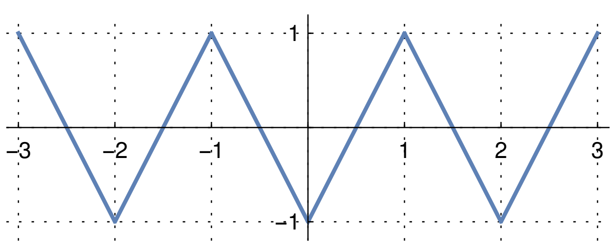
(a) Fourier series of $f(t)$ (b) Fourier cosine series of $f(t)$ (c) Fourier sine series of $f(t)$

In each sketch, carefully mark the values of the Fourier series at discontinuities.

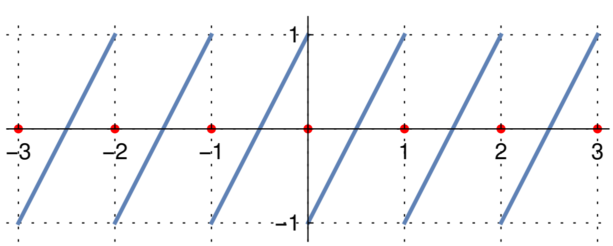
Solution.



(a)



(b)



(c)

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Problem 6. (7 points) Determine the equilibrium points of the system $\frac{dx}{dt} = xy - 3$, $\frac{dy}{dt} = x(y+1)$ and classify their stability.

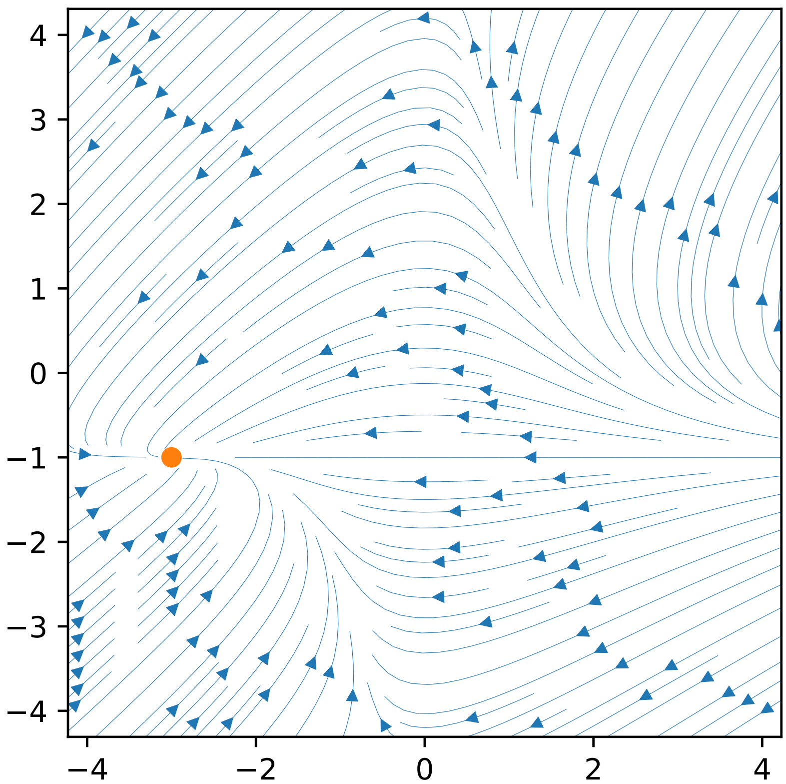
Solution. To find the equilibrium points, we solve $xy - 3 = 0$ and $x(y+1) = 0$. The second equation implies that we have $x = 0$ or $y = -1$. If $y = -1$, then the first equation implies $x = -3$. On the other hand, if $x = 0$, then the first equation has no solution. We conclude that the only equilibrium point is $(-3, -1)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} xy - 3 \\ x(y+1) \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y & x \\ y+1 & x \end{bmatrix}$.

At $(-3, -1)$, the Jacobian matrix is $J(-3,-1) = \begin{bmatrix} -1 & -3 \\ 0 & -3 \end{bmatrix}$. We can read off that the eigenvalues are $-1, -3$ (or we compute them as usual as roots of the characteristic polynomial $(-1-\lambda)(-3-\lambda) - 0$). Since both are negative, $(-3, -1)$ is a nodal sink. In particular, $(-3, -1)$ is asymptotically stable.

The following phase portrait confirms our analysis:



Problem 7. (5 points)

(a) By Euler's identity, we have $e^{ix} =$ $\boxed{}$.

(b) Determine the power series around $x = 0$: $\frac{5}{1+2x} =$ $\boxed{}$

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(c) Determine the power series around $x=0$: $3e^{-x} =$ [ ]

(d) Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$. How can we compute the a_n from $y(x)$? $a_n =$ [ ].

(e) If $f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{5}\right)$, then we can compute the b_n as $b_n =$ [ ].

Solution.

(a) $e^{ix} = \cos(x) + i\sin(x)$

(b) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{5}{1+2x} = 5\sum_{n=0}^{\infty} (-2x)^n = 5\sum_{n=0}^{\infty} (-2)^n x^n$.

(c) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $3e^{-x} = 3\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.

(d) $a_n = \frac{y^{(n)}(-1)}{n!}$ because this is the Taylor series of $f(x)$ around $x=-1$.

(e) This is the Fourier series of $f(t)$ (which has period 10 and so is $2L$-periodic with $L=5$) and it has the extra property that the a_n coefficients happen to be zero. The Fourier coefficients b_n can be computed as

$$b_n = \frac{1}{5}\int_{-5}^{5} f(t)\sin\left(\frac{n\pi t}{5}\right)\mathrm{d}t.$$

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(extra scratch paper)