Midterm #2 – Practice

MATH 332 — Differential Equations II
Midterm: Friday, Apr 11, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Determine the equilibrium points of the system $\frac{dx}{dt} = (x^2-4)y$, $\frac{dy}{dt} = x^2 y - 3xy + 5$ and classify their stability.

Problem 2. Let $y(x)$ be the unique solution to the IVP $y'' = x + 2y^3$, $y(0) = 1$, $y'(0) = 2$.

Determine the first several terms (up to x^4) in the power series of $y(x)$.

Problem 3. Consider the DE $y'' = x(x^2+7)\,y' + (x^2+3)y$.

Derive a recursive description of a power series solution $y(x)$ (around $x = 0$).

Problem 4. Find a minimum value for the radius of convergence of a power series solution to $(4x^2+1)y'' = \frac{3y'-y}{x+1}$ at $x = 3$.

Problem 5. Spell out the power series (around $x = 0$) of the following functions.

(a) e^{-3x}

(b) $\sin(3x^2)$

(c) $\dfrac{5}{1+7x^2}$

Problem 6.

(a) Suppose $y(x)$ has the power series $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$. How can we compute the a_n from $y(x)$?

(b) Suppose $f(t)$ has the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$.

How can we compute the a_n and b_n from $f(t)$?

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Problem 7. Consider the function $f(t) = 2(1 - t)$, defined for $t \in [0, 1]$.

(a) Sketch the Fourier series of $f(t)$ for $t \in [-4, 4]$.

(b) Sketch the Fourier cosine series of $f(t)$ for $t \in [-4, 4]$.

(c) Sketch the Fourier sine series of $f(t)$ for $t \in [-4, 4]$.

In each sketch, carefully mark the values of the Fourier series at discontinuities.

Problem 8. A mass-spring system is described by the DE $my'' + 7y = F(t)$ where $F(t)$ is an external force with period 3. For which values of m can resonance occur?

Problem 9. A mass-spring system is described by the equation

$$my'' + y = F(t),$$

where the external force has the Fourier series $F(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right)$.

(a) For which m does resonance occur?

(b) Find the general solution when $m = 1/9$.

Problem 10. Derive a recursive description of the power series (around $x = 0$) for $y(x) = \frac{1}{1 - 2x - 5x^2}$.

Problem 11. Compute the Fourier sine series of the function $f(t)$, defined for $t \in (0, L)$, with $f(t) = 3$.

Problem 12. Suppose that the matrix A satisfies $e^{Ax} = \frac{1}{7}\begin{bmatrix} e^{-9x} + 6e^{-2x} & -2e^{-9x} + 2e^{-2x} \\ -3e^{-9x} + 3e^{-2x} & 6e^{-9x} + e^{-2x} \end{bmatrix}$.

(a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 3e^x \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) What is A?

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