In the following example, we show how to find the special functions $u_n(x,t)$ using a technique called **separation of variables** that can be used to solve other simple partial differential equations as well.

Example 163. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u(0,t) &= u(L,t) = 0 && \text{(BC)} \\ u(x,0) &= f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We will first look for simple solutions of (PDE)+(BC) (and then we plan to take a combination of such solutions that satisfies (IC) as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called **separation of variables** and it is crucial for solving other PDEs as well.
- Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$.

  Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$. Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.
- Consider (BC). Note that $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$.

  [Because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution.]

  Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin(\frac{\pi n}{L}x)$ corresponding to the eigenvalues $\lambda = (\frac{\pi n}{L})^2$, $n = 1, 2, 3 \ldots$.
- On the other hand, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-(\frac{\pi n}{L})^2 k t}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \sin(\frac{\pi n}{L}x)$ solving (PDE)+(BC).
- We wish to combine these in such a way that (IC) holds as well.

  At $t = 0$, $u_n(x,0) = \sin(\frac{\pi n}{L}x)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an odd $2L$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n}{L}x)$. Note that

  $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\frac{\pi n}{L})^2 k t} \sin\left(\frac{\pi n}{L}x\right),$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$.

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Example 164. Find the unique solution $u(x,t)$ to: $\begin{array}{l} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1, \quad x \in (0,1) \end{array}$

Solution. This is the case $k=1$, $L=1$ and $f(x)=1$, $x \in (0,1)$, of Example 163.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 136. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x).$$

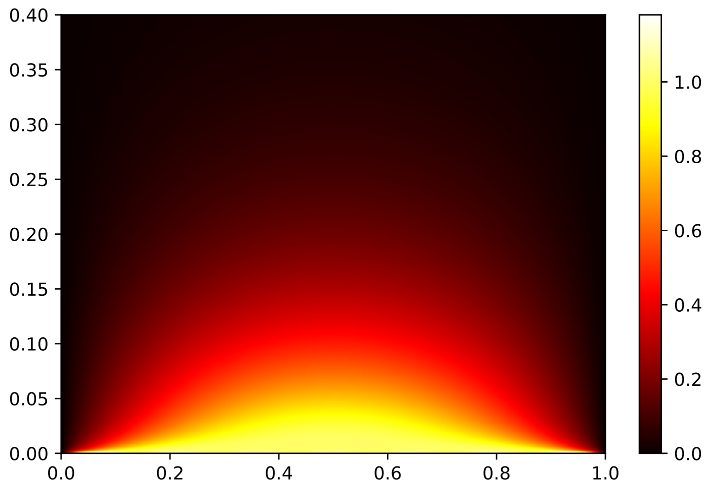
Hence, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n\pi x)$.

Comment. Note that, for $t>0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

We can use Sage to plot our solution using the terms $n = 1, 3, 5, \ldots, 19$ of the infinite sum:

```
>>> var('x,t');
>>> uxt = sum(4/(pi*n) * exp(-pi^2*n^2*t) * sin(pi*n*x) for n in range(1,20,2))
>>> density_plot(uxt, (x,0,1), (t,0,0.4), plot_points=200, cmap='hot')
```

The resulting plot should look similar to the following:



Can you make sense of the plot? Does that plot confirm our expectations?

[Note that the horizontal axis shows x for $x \in (0, 1)$, while the vertical axis shows t for $t \in (0, 0.4)$. Yellow represents 1 (for $t=0$, all values are 1 because of the initial condition), while black represents 0.]

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The boundary conditions in the next example model insulated ends.

Observe how we can proceed exactly as in Example 163. The main difference is that we need to find new functions $u_n(x,t)$ that solve the (same) PDE as well as the (different) boundary conditions.

Example 165. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x\in(0,L) && \text{(IC)}\end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).
  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda kT = 0$.
- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is shown in Example 159 that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{\pi n}{L}x\right)$ corresponding to $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3 \ldots$.
- On the other hand (as before), T solves $T' + \lambda kT = 0$, and hence $T(t) = e^{-\lambda kt} = e^{-\left(\frac{\pi n}{L}\right)^2 kt}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 kt}\cos\left(\frac{\pi n}{L}x\right)$ solving (PDE)+(BC).
- We wish to combine these in such a way that (IC) holds as well.
  At $t = 0$, $u_n(x,0) = \cos\left(\frac{\pi n}{L}x\right)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n}{L}x\right)$.
  Note that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \frac{a_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi n}{L}\right)^2 kt}\cos\left(\frac{\pi n}{L}x\right),$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$.

Example 166. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = 3u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(4,t) = 0 && \text{(BC)}\\ &u(x,0) = 2 + 5\cos(\pi x) - \cos(3\pi x), \;\; x\in(0,4) && \text{(IC)}\end{aligned}$$

Solution. This is the case $k = 3$, $L = 4$ that we solved in Example 165 where we found that the functions

$$u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 kt}\cos\left(\frac{\pi n}{L}x\right) = e^{-3\left(\frac{\pi n}{4}\right)^2 t}\cos\left(\frac{\pi n}{4}x\right)$$

solve (PDE)+(BC). Since $u_n(x,0) = \cos\left(\frac{\pi n}{4}x\right)$, we have

$$2u_0(x,0) + 5u_4(x,0) - u_{12}(x,0) = 2 + 5\cos(\pi x) - \cos(3\pi x),$$

which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = 2u_0(x,t) + 5u_4(x,t) - u_{12}(x,t) = 2 + 5e^{-3\pi^2 t}\cos(\pi x) - e^{-27\pi^2 t}\cos(3\pi x).$$