Boundary value problems

Example 154. The **IVP** (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$.

Initial value problems are often used when the problem depends on time. Then, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next example illustrates that such a boundary value problem (BVP) may or may not have a unique solution.

Example 155. Verify the following claims.

(a) The BVP $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

(b) The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

Solution.

(a) We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well.

(b) This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition is true for every B.

It is therefore natural to ask: for which λ does the BVP $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ have nonzero solutions? (We assume that $L > 0$.)

Such solutions are called **eigenfunctions** and λ is the corresponding **eigenvalue**.

Remark. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue.

Example 156. Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.

Such a problem is called an **eigenvalue problem**.

Solution. The solutions of the DE look different in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, so we consider them individually.

$\boldsymbol{\lambda = 0}$. Then $y(x) = Ax + B$ and $y(0) = y(L) = 0$ implies that $y(x) = 0$. No eigenfunction here.

$\boldsymbol{\lambda < 0}$. The roots of the characteristic polynomial are $\pm\sqrt{-\lambda}$. Writing $\rho = \sqrt{-\lambda}$, the general solution therefore is $y(x) = Ae^{\rho x} + Be^{-\rho x}$. $y(0) = A + B \overset{!}{=} 0$ implies $B = -A$. Using that, we get $y(L) = A(e^{\rho L} - e^{-\rho L}) \overset{!}{=} 0$. For eigenfunctions we need $A \neq 0$, so $e^{\rho L} = e^{-\rho L}$ which implies $\rho L = -\rho L$. This cannot happen since $\rho \neq 0$ and $L \neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda > 0}$. The roots of the characteristic polynomial are $\pm i\sqrt{\lambda}$. Writing $\rho = \sqrt{\lambda}$, the general solution thus is $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using that, $y(L) = B\sin(\rho L) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(\rho L) = 0$. This happens if $\rho L = n\pi$ for $n = 1, 2, 3, \ldots$ (since ρ and L are both positive, n must be positive as well). Equivalently, $\rho = \frac{n\pi}{L}$. Consequently, we find the eigenfunctions $y_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$, with eigenvalue $\lambda = \left(\frac{n\pi}{L}\right)^2$.

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Example 157. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0, L]$. The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy'' + Py = 0$, $y(0) = 0$, $y(L) = 0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$, with $\lambda = \frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda = \left(\frac{\pi n}{L}\right)^2$ for some $n = 1, 2, 3, \ldots$, means that buckling can only occur if $P = \left(\frac{\pi n}{L}\right)^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

https://en.wikipedia.org/wiki/Euler%27s_critical_load

Example 158. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

- $\boldsymbol{\lambda < 0}$. The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

- $\boldsymbol{\lambda = 0}$. The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

- $\boldsymbol{\lambda > 0}$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Example 159. Suppose $L > 0$. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0.$$

Solution. To solve this eigenvalue problem, we distinguish three cases:

- $\boldsymbol{\lambda < 0}$. Then, the roots are the real numbers $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y'(L) = A(Le^{Lr} - Le^{-Lr})$. Since $Le^{Lr} - Le^{-Lr} = 0$ only if $r = 0$, we see that $y'(L) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

- $\boldsymbol{\lambda = 0}$. Now, the general solution to the DE is $y(x) = A + Bx$. Then $y'(x) = B$ and we see that $y'(0) = 0$ and $y'(L) = 0$ if and only if $B = 0$. So $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y(x) = 1$.

- $\boldsymbol{\lambda > 0}$. Now, the roots are $\pm i\sqrt{\lambda}$ and $y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$. Hence, $y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. $y'(0) = B\sqrt{\lambda} = 0$ implies $B = 0$. Then, $y'(L) = -A\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0$ if and only if $\sin(L\sqrt{\lambda}) = 0$. The latter is true if and only if $L\sqrt{\lambda} = n\pi$ for some integer n. In that case, $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $y(x) = \cos\left(\frac{n\pi}{L}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3\ldots$, (why did we include $n = 0$ but excluded $n = -1, -2, \ldots$?!) with corresponding eigenfunctions $y(x) = \cos\left(\frac{\pi n}{L}x\right)$.