Hyperbolic sine and cosine

Review. Euler's formula states that $e^{it} = \cos(t) + i\sin(t)$.

Recall that a function $f(t)$ is **even** if $f(-t) = f(t)$. Likewise, it is **odd** if $f(-t) = -t$.

Note that $f(t) = t^n$ is even if and only if n is even. Likewise, $f(t) = t^n$ is odd if and only if n is odd. That's where the names are coming from.

Any function $f(t)$ can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that $f_{\text{even}}(t)$ indeed is even, and that $f_{\text{odd}}(t)$ indeed is an odd function (regardless of $f(t)$).

Example 148. The **hyperbolic cosine**, denoted $\cosh(t)$, is the even part of e^t. Likewise, the **hyperbolic sine**, denoted $\sinh(t)$, is the odd part of e^t.

- Equivalently, $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
- In particular, $e^t = \cosh(t) + \sinh(t)$.

  As recalled above, any function is the sum of its even and odd part.

  Comparing with Euler's formula, we find $\cosh(it) = \cos(t)$ and $\sinh(it) = i\sin(t)$. This indicates that $\cosh$ and $\sinh$ are related to $\cos$ and $\sin$, as their name suggests (see below for the "hyperbolic" part).
- $\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t)$.
- $\cosh(t)$ and $\sinh(t)$ both satisfy the DE $y'' = y$.

  We can write the general solution as $C_1 e^t + C_2 e^{-t}$ or, if useful, as $C_1\cosh(t) + C_2\sinh(t)$.
- $\cosh(t)^2 - \sinh(t)^2 = 1$

  Verify this by substituting $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

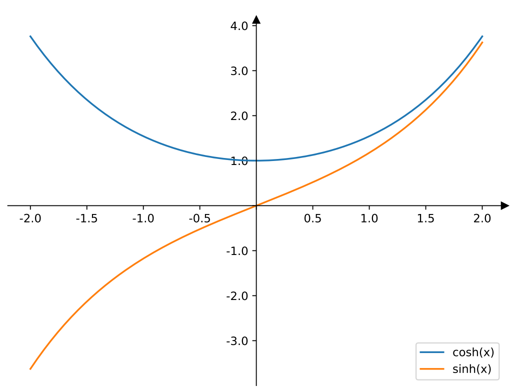
  Note that the equation $x^2 - y^2 = 1$ describes a **hyperbola** (just like $x^2 + y^2 = 1$ describes a circle).

  The above equation is saying that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$ is a parametrization of the hyperbola.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry

Homework. Write down the parallel properties of cosine and sine.



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Example 149. Write down a homogeneous linear differential equation satisfied by $y(x) = 5x^2 - 3\cosh(2x)$.

Comment. This is the same as finding an operator $p(D)$ such that $p(D)y = 0$.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, 0, 0, \pm 2$ (note that $\cosh(2x) = \frac{1}{2}(e^{2x} + e^{-2x})$ which contributes the roots ± 2).

Hence, the simplest differential equation is $D^3(D-2)(D+2)y = 0$.

Comment. This is an order 5 differential equation. If we wanted to, we could multiply out $D^3(D-2)(D+2) = D^3(D^2-4) = D^5 - 4D^3$ and write the differential equation in the "classical" form $y^{(5)} - 4y''' = 0$. However, there is typically no benefit in doing so because it is usually more useful to have the DE in factored form (so that the characteristic roots can just be read off). In general, only multiply out factored expressions if there is something to be gained from doing so!

Example 150. A homogeneous linear differential equation with constant coefficients is solved by $y(x) = 2e^{-7x}\cos(3x) - 5x\sinh(4x)$. Which characteristic roots must the DE have?

Solution. The characteristic roots of the differential equation must include $-7 \pm 3i, \pm 4, \pm 4$.

Example 151. Consider the DE $y'' - 2y' + y = 2x\sinh(3x) + 7x^2$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $\pm 3, \pm 3, 0, 0, 0$. Hence, there has to be a particular solution of the form $y_p = (A_1 + A_2 x)\cosh(3x) + (A_3 + A_4 x)\sinh(3x) + A_5 + A_6 x + A_7 x^2$.

(We can then plug into the DE to determine the (unique) values of the coefficients $A_1, A_2, \ldots, A_7$.)

Comment. If we prefer, we can, of course, also express $\sinh(3x)$ in terms of exponentials. Then the DE becomes $y'' - 2y' + y = xe^{3x} - xe^{-3x} + 7x^2$. The characteristic roots of the DE remain the same. The simplest form of a particular solution now is $y_p = (B_1 + B_2 x)e^{3x} + (B_3 + B_4 x)e^{-3x} + B_5 + B_6 x + B_7 x^2$. Make sure that you see that this is equivalent to our earlier form using $\cosh$ and $\sinh$.