Review: the motion of a mass on a spring

The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -ky$ combined with Newton's second law $F = ma = my''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$. Equivalently, its (circular) **frequency** is ω.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ (can you explain/prove this?) where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.

More generally, the motion of a mass m on a spring, with damping and with an external force $f(t)$ taken into account, can be modeled by the DE

$$my'' + dy' + ky = f(t).$$

Note that each term is representing a force: $my'' = ma$ is the force due to Newton's second law ($F = ma$), the term dy' models damping (proportional to the velocity), the term ky represents the force due to Hooke's law, and the term $f(t)$ represents an external force that acts on the mass at time t.

Fourier series and linear differential equations

In the following examples, we consider inhomogeneous linear DEs $p(D)y = F(t)$ where $F(t)$ is a periodic function that can be expressed as a Fourier series. We first review the notion of **resonance** (and how to predict it) and then solve such DEs.

Example 142. Consider the linear DE $my'' + ky = \cos(\omega t)$. For which (external) **frequencies** $\omega > 0$ does **resonance** occur?

Solution. The characteristic roots (the roots of $p(D) = mD^2 + k$) are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my'' + ky = 0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega = \omega_0$ when the external frequency matches the natural frequency.

Review. If $\omega \neq \omega_0$ (overlapping roots), then there is particular solution of the form $y_p(t) = A\cos(\omega t) + B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t) + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega = \omega_0$, then the general solution is $y(t) = (C_1 + At)\cos(\omega_0 t) + (C_2 + Bt)\sin(\omega_0 t)$ and this function is unbounded.

Example 143. A mass-spring system is described by the DE $2y'' + 32y = \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2+1}$.

For which ω does resonance occur?

Solution. The roots of $p(D) = 2D^2 + 32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $\omega = 4/n$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 144. A mass-spring system is described by the DE $my'' + y = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for some $n \in \{1, 2, 3, \ldots\}$.

Example 145. A mass-spring system is described by the DE $3y'' + ky = F(t)$ where $F(t)$ is an external force with period 5. For which values of k can resonance occur?

Solution. $F(t)$ has a Fourier series of the form $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\cos\left(\frac{2\pi n t}{5}\right) + b_n\sin\left(\frac{2\pi n t}{5}\right)\right)$.

The roots of $p(D) = 3D^2 + k$ are $\pm i\sqrt{\frac{k}{3}}$, so that the natural frequency is $\sqrt{\frac{k}{3}}$. Resonance therefore can occur if $\sqrt{\frac{k}{3}} = \frac{2\pi n}{5}$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance can occur if $k = \frac{12\pi^2 n^2}{25}$ for some $n \in \{1, 2, 3, \ldots\}$.

Note. Resonance will occur for $k = \frac{12\pi^2 n^2}{25}$ unless both of the corresponding Fourier coefficients a_n and b_n are 0.

Note. The term $a_0/2$ in $F(t)$ corresponds to a characteristic root of 0 and cannot lead to resonance.

Though it requires some effort, we already know how to solve $p(D)y = F(t)$ for periodic forces $F(t)$, once we have a Fourier series for $F(t)$.

The same approach works for linear differential equations of higher order, or even systems of equations.

Example 146. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10\sum_{n \text{ odd}} \frac{4}{\pi n}\sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2 y_p$ (can you see why without computing two derivatives?), we get

$$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \overset{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.

- We combine the particular solutions found in the previous step, to see that

$$2y'' + 32y = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(\pi n t) \quad \text{is solved by} \quad y_p = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Example 147. Find a particular solution of $2y''+32y=F(t)$, with $F(t)$ the 2π-periodic function such that $F(t)=10t$ for $t\in(-\pi,\pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{20}{n}\sin(nt)$. [Exercise!]
- We next solve the equation $2y''+32y=\sin(nt)$ for $n=1,2,3,\ldots$. Note, however, that **resonance** occurs for $n=4$, so we need to treat that case separately. If $n\neq 4$ then we find, as in the previous example, that $y_p(t)=\frac{\sin(nt)}{32-2n^2}$. [Note how this fails for $n=4$!]

  For $2y''+32y=\sin(4t)$, we begin with $y_p=At\cos(4t)+Bt\sin(4t)$. Then $y_p'=(A+4Bt)\cos(4t)+(B-4At)\sin(4t)$, and $y_p''=(8B-16At)\cos(4t)+(-8A-16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p''+32y_p=16B\cos(4t)-16A\sin(4t)\overset{!}{=}\sin(4t)$, and thus $B=0$, $A=-\frac{1}{16}$. So, $y_p=-\frac{1}{16}t\cos(4t)$.
- We combine the particular solutions to get that our DE

$$2y''+32y=-5\sin(4t)+\sum_{\substack{n=1\\n\neq 4}}^{\infty}(-1)^{n+1}\frac{20}{n}\sin(nt)$$

  is solved by

$$y_p(t)=\frac{5}{16}t\cos(4t)+\sum_{\substack{n=1\\n\neq 4}}^{\infty}(-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32-2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

Important comment. Note that the general solution is

$$y(t)=\frac{5}{16}t\cos(4t)+\sum_{\substack{n=1\\n\neq 4}}^{\infty}(-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32-2n^2}+C_1\cos(4t)+C_2\sin(4t)$$

and that it always features the resonant term.

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