Fourier series

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions 1, $\cos(t)$, $\sin(t)$, $\cos(2t)$, $\sin(2t)$, $\ldots$ are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 130. Every* 2π-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity of f, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i \sin(nt)$.

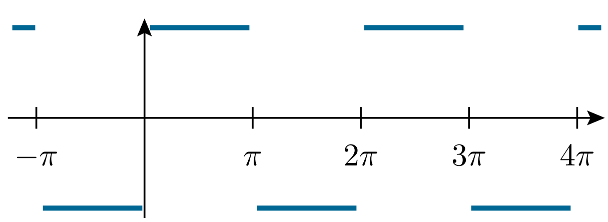
Definition 131. Let $L > 0$. $f(t)$ is ***L*-periodic** if $f(t + L) = f(t)$ for all t. The smallest such L is called the **(fundamental) period** of f.

Example 132. The fundamental period of $\cos(nt)$ is $2\pi / n$.

Example 133. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π-periodic for every integer n. And so are their linear combinations. (Thus, 2π-periodic functions form a vector space!)

Example 134. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$

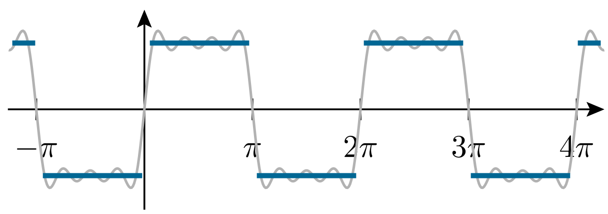


Armin Straub
straub@southalabama.edu

Solution. We compute $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t + \int_{0}^{\pi}\cos(nt)\mathrm{d}t\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t + \int_{0}^{\pi}\sin(nt)\mathrm{d}t\right] = \frac{2}{\pi n}[1-\cos(n\pi)] \\ &= \frac{2}{\pi n}[1-(-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: https://en.wikipedia.org/wiki/Gibbs_phenomenon

Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not completely obvious. (Do you recall, for instance, the alternating sign test from Calculus II?) For instance, the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges.

Fourier series with general period

The case of 2π-periodic functions generalizes easily to the case of general periodic functions.

Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

Theorem 135. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Armin Straub
straub@southalabama.edu

Comment. This follows from Theorem 130 because, if $f(t)$ has period $2L$, then $\tilde{f}(t) := f(Lt/\pi)$ has period 2π.

Example 136. Find the Fourier series of the 2-periodic function $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t) = f(\pi t)$, with f as in the previous example, to get $g(t) = f(\pi t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t)$.