Once we have a power series solution $y(x)$, a natural question is: for which x does the series converge?

Recall. A power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a **radius of convergence** R.
The series converges for all x with $|x - x_0| < R$ and it diverges for all x with $|x - x_0| > R$.

Definition 108. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$. x_0 is called an **ordinary point** if the coefficients $p_j(x)$, as well as $f(x)$, are analytic at $x = x_0$. Otherwise, x_0 is called a **singular point**.

Example 109. Determine the singular points of $(x+2)y'' - x^2 y' + 3y = 0$.

Solution. Rewriting the DE as $y'' - \frac{x^2}{x+2} y' + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.

Example 110. Determine the singular points of $(x^2+1) y''' = \frac{y}{x-5}$.

Solution. Rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.

Theorem 111. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$. Suppose that x_0 is an ordinary point and that R is the distance to the closest singular point. Then any IVP specifying $y(x_0)$, $y'(x_0)$, ..., $y^{(n-1)}(x_0)$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and that series has radius of convergence at least R.

In particular. The DE has a general solution consisting of n solutions $y(x)$ that are analytic at $x = x_0$.

Comment. Most textbooks only discuss the case of 2nd order DEs. For a discussion of the higher order case (in terms of first order systems!) see, for instance, Chapter 4.5 in *Ordinary Differential Equations* by N. Lebovitz. The book is freely available at: `http://people.cs.uchicago.edu/~lebovitz/odes.html`

Example 112. Find a minimum value for the radius of convergence of a power series solution to $(x+2)y'' - x^2 y' + 3y = 0$ at $x = 3$.

Solution. As before, rewriting the DE as $y'' - \frac{x^2}{x+2} y' + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.
Note that $x = 3$ is an ordinary point of the DE and that the distance to the singular point is $|3 - (-2)| = 5$.
Hence, the DE has power series solutions about $x = 3$ with radius of convergence at least 5.

Example 113. Find a minimum value for the radius of convergence of a power series solution to $(x^2+1) y''' = \frac{y}{x-5}$ at $x = 2$.

Solution. As before, rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.
Note that $x = 2$ is an ordinary point of the DE and that the distance to the nearest singular point is $|2 - i| = \sqrt{5}$ (the distances are $|2 - 5| = 3$, $|2 - i| = |2 - (-i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$).
Hence, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{5}$.

Example 114. (Airy equation, once more) Let $y(x)$ be the solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Earlier, we determined the power series of $y(x)$. What is its radius of convergence?

Solution. $y'' = xy$ has no singular points. Hence, the radius of convergence is ∞. (In other words, the power series converges for all x.)

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