Application of variation of constants: the second order case

Review. In Theorem 96 we showed that $\boldsymbol{y}' = A(t)\,\boldsymbol{y} + \boldsymbol{f}(t)$ has the particular solution

$$\boldsymbol{y}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t,$$

where $\Phi(t)$ is a fundamental matrix solution to $\boldsymbol{y}' = A(t)\,\boldsymbol{y}$.

Let us apply this result to the case of a second-order LDE

$$y'' + P(t)y' + Q(t)y = F(t).$$

We can write this DE as a first-order system by introducing the vector $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -Q(t) & -P(t) \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

If the second-order homogeneous DE (that is, $y'' + P(t)y' + Q(t)y = 0$) has general solution $C_1y_1(t) + C_2y_2(t)$, then a fundamentral matrix for the first-order homogeneous system is

$$\Phi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

(recall that each column of $\Phi(t)$ represents a solution $\boldsymbol{y}$ of the system). Our formula from Theorem 96 now gives us a particular solution of the inhomogeneous system:

$$\begin{aligned}
\boldsymbol{y}_p(t) &= \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t \\
&= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\int \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ F \end{bmatrix}\mathrm{d}t \\
&= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\int \frac{F}{y_1y_2' - y_1'y_2}\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}\mathrm{d}t \\
&= \int \frac{-y_2F}{y_1y_2' - y_1'y_2}\mathrm{d}t\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + \int \frac{y_1F}{y_1y_2' - y_1'y_2}\mathrm{d}t\begin{bmatrix} y_2 \\ y_2' \end{bmatrix}
\end{aligned}$$

The first entry of $\boldsymbol{y}_p$ is the corresponding particular solution to the second-order inhomogeneous DE:

$$y_p(t) = C_1(t)y_1(t) + C_2(t)y_2(t), \quad C_1(t) = \int \frac{-y_2(t)F(t)}{W(t)}\mathrm{d}t, \quad C_2(t) = \int \frac{y_1(t)F(t)}{W(t)}\mathrm{d}t.$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the **Wronskian**.

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Some special functions and the power series method

Review: power series

Definition 100. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  We can rewrite the series as $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$.

  The result is a power series just like the one we started with. Likewise, for higher derivatives.

- $\int y(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 101. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

In particular, if $y(x)$ is analytic at $x = 0$, then

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = y(0) + y'(0)x + \frac{1}{2} y''(0) x^2 + \frac{1}{6} y'''(0) x^3 + \ldots$$

We have already seen the following example.

Example 102. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$ (as well as $y(0) = 1$).

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \ldots$

Example 103. Determine the power series for $7e^{3x}$ (at $x = 0$).

Solution. Instead of starting from scratch, we can use that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to conclude that

$$7e^{3x} = 7 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{n!} x^n = 7 + 21x + \frac{63}{2} x^2 + \frac{63}{2} x^3 + \frac{189}{8} x^4 + \ldots$$

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Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 104. (Airy equation, part I) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \frac{1}{720}y^{(6)}(0)x^6 + \ldots$
$= a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ becomes $y = a + bx + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial conditions.

To determine a_2, a_3, a_4, a_5, a_6, we equate the coefficients of:

$$\begin{aligned} y'' &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \ldots \\ xy &= ax + bx^2 + a_2x^3 + a_3x^4 + \ldots \end{aligned}$$

We find $2a_2 = 0$, $6a_3 = a$, $12a_4 = b$, $20a_5 = a_2$, $30a_6 = a_3$.

So $a_2 = 0$, $a_3 = \frac{a}{6}$, $a_4 = \frac{b}{12}$, $a_5 = \frac{a_2}{20} = 0$, $a_6 = \frac{a_3}{30} = \frac{a}{180}$. Hence, $y(x) = a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$