Application: Mixing problems

Example 91. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

We wish to understand how much salt is in the tanks after t minutes.

(a) Derive a system of differential equations.

(b) Determine the equilibrium points and classify their stability. What does this mean here?

(c) Solve the system to find explicit formulas for how much salt is in the tanks after t minutes.

Solution.

(a) Note that the amount of water in each tank is constant because the flows balance each other.

Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t+\Delta t]$:

$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.

$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\frac{d}{dt}\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

(b) Note that this system is autonomous! Otherwise, we could not pursue our stability analysis.

To find the equilibrium point (since the system is linear, there should be just one), we set $\frac{d}{dt}\boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We find $\boldsymbol{y} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}^{-1}\begin{bmatrix} -27 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

The characteristic polynomial of $\begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$ is $(-3-\lambda)(-8-\lambda) - 6 = \lambda^2 + 11\lambda + 18 = (\lambda+9)(\lambda+2)$.

Hence, the eigenvalues are $-9, -2$. Since they are both negative, the equilibrium point is a nodal sink and, in particular, asymptotically stable.

Having an equilibrium point at $(12, 4.5)$, means that, if the salt amounts are $y_1 = 12$, $y_2 = 4.5$, then they won't change over time (but will remain unchanged at these levels). The fact that it is asymptotically stable means that salt amounts close to these balanced levels will, over time, approach the equilibrium levels. (Because the system is linear, this is also true for levels that are not "close".)

We could have "seen" the equilibrium point!

Indeed, noticing that, for a constant (equilibrium) particular solution $\boldsymbol{y}$, each tank has to have a constant concentration of 0.5lb/gal of salt, we find directly $\boldsymbol{y} = 0.5\begin{bmatrix} 24 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

(c) We sketch the computation. From the previous part, we know that a particular solution is $\boldsymbol{y}_p = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

A fundamental matrix of the homogeneous system is $\Phi(t) = \begin{bmatrix} e^{-9t} & 2e^{-2t} \\ -3e^{-9t} & e^{-2t} \end{bmatrix}$ (compute eigenvectors!).

Hence, the general solution to the inhomogeneous system is $\boldsymbol{y}(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix} + C_1\begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{-9t} + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-2t}$.

Using the initial condition $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, we get the equation $\begin{bmatrix} 12 \\ 4.5 \end{bmatrix} + C_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solving this, we find $C_1 = 0$ and $C_2 = -4.5$.

In conclusion, the unique solution to the IVP is $\boldsymbol{y}(t) = \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$.

Comment. We will soon discuss inhomogeneous linear systems in general.

Example 92. Consider two brine tanks. Initially, tank T_1 is filled with 10gal water containing 2lb salt, and tank T_2 with 5gal pure water.

- T_1 is being filled with 4gal/min water containing 0.5lb/gal salt.
- 5gal/min well-mixed solution flows out of T_1 into T_2.
- 2gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 1gal/min well-mixed solution is leaving T_2.

Derive a system of equations for the amount of salt in the tanks after t minutes.

Solution. Let $V_i(t)$ denote the amount of solution (in gal) in tank T_i after time t (in min). Then $V_1(t) = 10 + 4t - 5t + 2t = 10 + t$ while $V_2(t) = 5 + 5t - 2t - t = 5 + 2t$.

Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 4 \cdot \frac{1}{2} \cdot \Delta t - 5 \cdot \frac{y_1}{V_1} \cdot \Delta t + 2 \cdot \frac{y_2}{V_2} \cdot \Delta t$, so $y_1' = 2 - 5\frac{y_1}{V_1} + 2\frac{y_2}{V_2}$. Also, $y_1(0) = 2$.

$\Delta y_2 \approx 5 \cdot \frac{y_1}{V_1} \cdot \Delta t - (2 + 1) \cdot \frac{y_2}{V_2} \cdot \Delta t$, so $y_2' = 5\frac{y_1}{V_1} - 3\frac{y_2}{V_2}$. Also, $y_2(0) = 0$.

In conclusion, we have obtained the system of equations

$$\begin{aligned} y_1' &= -\frac{5}{10+t} y_1 + \frac{2}{5+2t} y_2 + 2, \qquad && y_1(0) = 2, \\ y_2' &= \frac{5}{10+t} y_1 - \frac{3}{5+2t} y_2, && y_2(0) = 0. \end{aligned}$$

Note that this is a system of linear DEs. It is inhomogeneous (because of the $+2$ in the first equation). Its coefficients are not constant. As a consequence, this system is not autonomous and so we cannot apply our stability analysis.

In matrix-vector form. If we write $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then the system becomes

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{5}{10+t} & \frac{2}{5+2t} \\ \frac{5}{10+t} & -\frac{3}{5+2t} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Armin Straub
straub@southalabama.edu