Notes for Lecture 15

Wed, 2/26/2025

Review: Linearizations of nonlinear functions (cont'd)

Review.

Recall from Calculus I that a function $f(x)$ around a point x_0 has the linearization

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Recall from Calculus III that a function $f(x, y)$ around a point (x_0, y_0) has the linearization

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

It follows that a vector function $\boldsymbol{f}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$ around a point (x_0, y_0) has the linearization

$$\begin{aligned} \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} &\approx \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0, y_0) \\ g_x(x_0, y_0) \end{bmatrix}(x - x_0) + \begin{bmatrix} f_y(x_0, y_0) \\ g_y(x_0, y_0) \end{bmatrix}(y - y_0) \\ &= \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} + \underbrace{\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}}_{= J(x_0, y_0)} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \end{aligned}$$

The matrix $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ is called the **Jacobian matrix** of $\boldsymbol{f}(x, y)$.

Example 85. Determine the linearization of the vector function $\begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$ at $(2, 1)$.

Solution. If $\begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} = \begin{bmatrix} 3 + 2xy^2 \\ x(y^3 - 2x) \end{bmatrix}$, then the Jacobian matrix is

$$J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2y^2 & 4xy \\ y^3 - 4x & 3xy^2 \end{bmatrix}.$$

In particular, $J(2, 1) = \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix}$. The linearization is $\begin{bmatrix} f(2, 1) \\ g(2, 1) \end{bmatrix} + J(2, 1) \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix}$.

Important comment. If we multiply out the matrix-vector product, then we get $\begin{bmatrix} 7 + 2(x - 2) + 8(y - 1) \\ -6 - 7(x - 2) + 6(y - 1) \end{bmatrix}$.

In the first component we get exactly what we got for the linearization of $f(x, y)$ in the previous example. Likewise, the second component is the linearization of $g(x, y)$ by itself.

Armin Straub
straub@southalabama.edu

Stability of nonlinear autonomous systems

We now observe that we can (typically) determine the stability of an equilibrium point of a nonlinear system by simply linearizing at that point.

(stability of autonomous nonlinear 2-dimensional systems)

Suppose that (x_0, y_0) is an equilibrium point of the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$.

If the Jacobian matrix $J(x_0, y_0)$ is invertible, then its eigenvalues determine the stability and behaviour of the equlibrium point as for a linear system except in the following cases:

- If the eigenvalues are pure imaginary, we cannot predict stability (the equilibrium point could be either a center or a spiral source/sink; whereas the equilibrium point of the linearization is a center).
- If the eigenvalues are real and equal, then the equilibrium point could be either nodal or spiral (whereas the linearization has a nodal equilibrium point). The stability, however, is the same.

Comment. We need the Jacobian matrix $J(x_0, y_0)$ to be invertible, so that the linearized system has a unique equilibrium point.

Plot, for instance, the phase portrait of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x-2y)x \\ (x-2)y \end{bmatrix}$.

Purely imaginary eigenvalues? The issue with pure imaginary eigenvalues here comes from the fact that the linearization is only an approximation, with the true (nonlinear) behaviour slightly deviating. Slightly perturbing purely imaginary roots can also lead to (small but) positive real part (unstable; spiral source) or negative real part (asymptotically stable; spiral sink).

Real repeated eigenvalue? The issue with a real repeated eigenvalue is similar. Slightly perturbing such a root can lead to real eigenvalues (nodal) or a pair of complex conjugate eigenvalues (spiral). However, the real part of these perturbations still has the same sign so that we can still predict the stability itself.

The following is a continuation of Example 77:

Example 86. (cont'd) Consider again the system $\frac{dx}{dt} = x \cdot (y-1)$, $\frac{dy}{dt} = y \cdot (x-1)$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. See Example 77 for the phase portrait. However, we will not use it in the following.

To find the equilibrium points, we solve $x(y-1)=0$ (that is, $x=0$ or $y=1$) and $y(x-1)=0$ (that is, $x=1$ or $y=0$). We conclude that the equilibrium points are $(0,0)$ and $(1,1)$.

Our system is $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ with $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x\cdot(y-1) \\ y\cdot(x-1) \end{bmatrix}$.

The Jacobian matrix is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y-1 & x \\ y & x-1 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We can read off that the eigenvalues are $-1, -1$. Since they are both negative, $(0,0)$ is asymptotically stable.

  Since this is a real repeated eigenvalue, we cannot immediately tell whether $(0,0)$ is a nodal sink (it is a nodal sink for the linearization!) or a nodal spiral. (Since our system is nonlinear, the linearization is just an approximation. Similarly, you can think of the eigenvalues $-1, -1$ as being somewhat approximate. Slight jiggling could change them to something like $-1 \pm 0.001i$ which would correspond to a nodal spiral.)

- At $(1,1)$, the Jacobian matrix is $J(1,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1$, which has roots ± 1. These are the eigenvalues. Since one is positive and the other is negative, $(1,1)$ is a saddle. In particular, $(1,1)$ is unstable.

Example 87. Consider again the system $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$. Without consulting a plot, determine the equilibrium points and classify their stability.

Solution. To find the equilibrium points, we solve $2y - x^2 = 0$ and $(x-3)(x-y)$.
It follows from the second equation that $x = 3$ or $x = y$:

- If $x = 3$, then the first equation implies $y = \frac{9}{2}$.
- If $x = y$, then the first equation becomes $2y - y^2 = 0$, which has solutions $y = 0$ and $y = 2$.

Hence, the equilibrium points are $(0,0)$, $(2,2)$ and $\left(3, \frac{9}{2}\right)$.

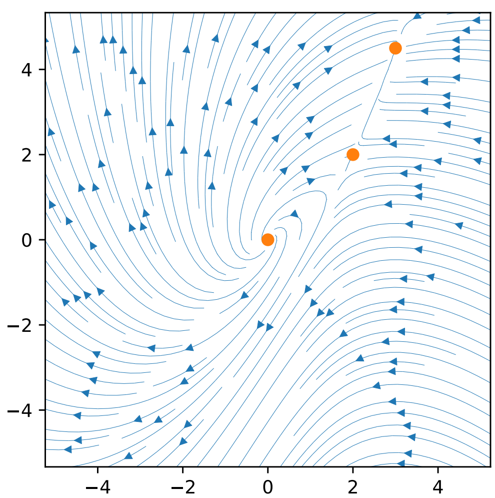
The Jacobian matrix of $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2y - x^2 \\ (x-3)(x-y) \end{bmatrix}$ is $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -2x & 2 \\ 2x - y - 3 & -x + 3 \end{bmatrix}$.

- At $(0,0)$, the Jacobian matrix is $J = \begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(3 \pm i\sqrt{15})$.

  Since these are complex with positive real part, $(0,0)$ is a spiral source and, in particular, unstable.
- At $(2,2)$, the Jacobian matrix is $J = \begin{bmatrix} -4 & 2 \\ -1 & 1 \end{bmatrix}$. The eigenvalues are $\frac{1}{2}(-3 \pm \sqrt{17}) \approx -3.562, 0.562$.

  Since these are real with opposite signs, $(2,2)$ is a saddle and, in particular, unstable.
- At $\left(3, \frac{9}{2}\right)$, the Jacobian matrix is $J = \begin{bmatrix} -6 & 2 \\ -\frac{3}{2} & 0 \end{bmatrix}$. The eigenvalues are $-3 \pm \sqrt{6} \approx -5.449, -0.551$.

  Since these are real and both negative, $\left(3, \frac{9}{2}\right)$ is a nodal sink and, in particular, asymptotically stable.



Comment. Can you confirm our analysis in the above plot? Note that it is becoming hard to see the details. One solution would be to make separate phase portraits focusing on the vicinity of each equilibrium plot. Do it!

Armin Straub
straub@southalabama.edu