Sketch of Lecture 32

Example 137. Find the unique solution $u(x,t)$ to: $\begin{array}{l} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1, \quad x \in (0,1) \end{array}$

Solution. This is the case $k=1$, $L=1$ and $f(x)=1$, $x \in (0,1)$, of the previous example.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 116. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x).$$

Hence, $u(x,t) = \displaystyle\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n\pi x).$

Comment. Note that, for $t > 0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

Make some 3D plots!

Armin Straub
straub@southalabama.edu

The boundary conditions in the next example model insulated ends.

Example 138. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).
  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.

- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is left as a homework to show that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{\pi n}{L}x\right)$ corresponding to $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3 \ldots$. [See practice problems.]

- On the other hand (as before), T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L}x\right)$ solving (PDE)+(BC).

- We wish to combine these in such a way that (IC) holds.
  At $t = 0$, $u_n(x,0) = \cos\left(\frac{\pi n}{L}x\right)$. All of these are $2L$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n}{L}x\right)$.
  Note that
  $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$
  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by
$$u(x,t) = \frac{a_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} a_n\, u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, e^{-\left(\frac{\pi n}{L}\right)^2 k t}\cos\left(\frac{\pi n}{L}x\right),$$
where
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$