Example 131. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

$\lambda < 0$. The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

$\lambda = 0$. The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

$\lambda > 0$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Partial differential equations

The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x,t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0, L]$.

Notation. $u(x,t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t}u$ and $u_{xx} = \frac{\partial^2}{\partial x^2}u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = ku_{xx}$, with $k > 0$.

(heat equation)

$$u_t = ku_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1u_1 + c_2u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial/\partial x, \partial/\partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

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Example 132. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 133. To get a feeling, let us find some other solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- ... to be continued ...
  Can you find further solutions?

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $u(x,0) = f(x)$ (IC)
  This specifies an initial temperature distribution at time $t = 0$.
- **Boundary condition** at $x = 0$ and $x = L$: (BC)
  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:
  - $u(0,t) = A$, $u(L,t) = B$
    This models a rod where one end is kept at temperature A and the other end at temperature B.
  - $u_x(0,t) = u_x(L,t) = 0$
    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0,t) = A$, $u(L,t) = B$ into $u(0,t) = u(L,t) = 0$ by using the fact that $u(t,x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?

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