Example 102. Derive a recursive description of the power series for $y(x) = \frac{1+7x}{1-x-2x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$1+7x = (1-x-2x^2)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 2\sum_{n=2}^{\infty} a_{n-2} x^n.$$

We compare coefficients of x^n:

- $n=0$: $1 = a_0$.
- $n=1$: $7 = a_1 - a_0$, so that $a_1 = 7 + a_0 = 8$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - 2a_{n-2}$.

If we prefer, we can rewrite the final recurrence as $a_{n+2} - a_{n+1} - 2a_n = 0$ for $n \geqslant 0$. The initial conditions are $a_0 = 1$, $a_1 = 8$.

Comment. In terms of the recurrence operator N, the recurrence is $(N^2 - N - 2)a_n = 0$.

Comment. As in Example 54, we can solve this recurrence and obtain a Binet-like formula for a_n. In this particular case, we find $a_n = 3 \cdot 2^n - 2(-1)^n$.

Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x = 0$.)

Example 103. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. Determine its power series.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix) = \cos(x)$ (because $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)$ is similarly defined to be the odd part of e^x.

Example 104. **(geometric series)** Determine $y(x) = \sum_{n=0}^{\infty} x^n$.

Solution. Note that $xy = y - 1$. Hence, $y = \frac{1}{1-x}$.

Comment. The radius of convergence of this series is 1. This is easy to see directly. But note that it also follows from Theorem 92 since $y(x)$ solves the "differential" (inhomogeneous) equation $(1-x)y = 1$, for which the only singular point is $x = 1$.

Example 105. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to get $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

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Example 106. (extra) Determine a power series for $\ln(x)$ around $x=1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x=0$ (see the final step).

Observe that $\ln(x+1) = \int \frac{\mathrm{d}x}{1+x} + C$ and that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

Integrating, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$. Since $\ln(1)=0$, we conclude that $C=0$.

Finally, $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+1}$.

Comment. Choosing $x=2$ in $\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+1}$ results in $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.
The latter is the alternating harmonic sum.
Can you see from here why the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges?

Example 107. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x) = \int \frac{\mathrm{d}x}{1+x^2} + C$. Hence, we need to integrate $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

It follows that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$. Since $\arctan(0) = 0$, we conclude that $C=0$.

Example 108. (error function) Determine a power series for $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 109. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x) = \tan(x)$ is the unique solution to the IVP $y' = 1 + y^2$, $y(0) = 0$.
We can therefore proceed to determine the first few power series coefficients as we did earlier.
That is, we plug $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ into the DE. Note that $y(0) = 0$ means $a_0 = 0$.

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$

$1 + y^2 = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + \ldots)^2 = 1 + a_1^2 x^2 + (2a_1 a_2) x^3 + (2a_1 a_3 + a_2^2) x^4 + \ldots$

Comparing coefficients, we find: $a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1^2$, $4a_4 = 2a_1 a_2$, $5a_5 = 2a_1 a_3 + a_2^2$.

Solving for $a_2, a_3, \ldots$, we conclude that $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n = 0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.
The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y' = 1 + y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There is no analog of Theorem 92.)