

Crash course: Eigenvalues and eigenvectors

If $A\mathbf{x} = \lambda\mathbf{x}$ (and $\mathbf{x} \neq \mathbf{0}$), then \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\mathbf{x} = \lambda\mathbf{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ \iff A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ \iff (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

This homogeneous system has a nontrivial solution \mathbf{x} if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A :

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ , called the **characteristic polynomial** of A .

(b) Then, for each eigenvalue λ , find corresponding eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example 52. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8 - \lambda & -10 \\ 5 & -7 - \lambda \end{bmatrix}\right) = (8 - \lambda)(-7 - \lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix}\mathbf{x} = \mathbf{0}$.
Hence, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.
- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$.
Hence, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 53. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.