

Solving recurrence equations

Example 45. Find the general solution to the recursion $a_{n+2} = a_{n+1} + 6a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6 = (N - 3)(N + 2)$.

Since $(N - 3)a_n = 0$ has solution $a_n = C \cdot 3^n$, and since $(N + 2)a_n = 0$ has solution $a_n = C \cdot (-2)^n$ (compare previous example), we conclude that the general solution is $a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$.

Comment. This must indeed be the general solution, because the two degrees of freedom C_1, C_2 allow us to match any initial conditions $a_0 = A, a_1 = B$: the two equations $C_1 + C_2 = A$ and $3C_1 - 2C_2 = B$ in matrix form are $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$, which always has a (unique) solution because $\det\left(\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\right) = -5 \neq 0$.

Example 46. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots 2, 1, -1.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Theorem 47. (Binet's formula) $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2 . We can do that using the two initial conditions: $F_0 = C_1 + C_2 = 0, F_1 = C_1 \cdot \frac{1 + \sqrt{5}}{2} + C_2 \cdot \frac{1 - \sqrt{5}}{2} = 1$.

Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. \square

Comment. For large n , $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n , it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \xrightarrow{n \rightarrow \infty} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 48. Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots 2, 2.

So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.

Comment. This is analogous to $(D - 2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.

Check! Let's check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N - 2)^2 a_n = 0$.

$(N - 2)n \cdot 2^n = (n + 1)2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N - 2)^2 n \cdot 2^n = (N - 2)2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 20 for recurrence equations (RE):

Solutions to such recurrences are called **C-finite sequences**.

Theorem 49. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \dots, k - 1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ equals the largest root r that contributes to a_n .

Example 50. Find the general solution to the recursion $a_{n+3} = 3a_{n+2} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 3N^2 + 4$ has roots $2, 2, -1$. (Here, we used some help from a computer algebra system to find the roots.)

Hence, the general solution is $a_n = (C_1 + C_2 n) \cdot 2^n + C_3 \cdot (-1)^n$.

Example 51. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. 0, 1, 2, 8, 24, 80, 256, 832, ...

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Solution. Proceeding as for the Fibonacci numbers, we find $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2\sqrt{5}}$.