Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Homogeneous linear DEs with constant coefficients

Example 17. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2) y = 0$.
Note that $D^2 - D - 2 = (D - 2)(D + 1)$ is the **characteristic polynomial**.
It follows that we get solutions to $(D - 2)(D + 1) y = 0$ from $(D - 2) y = 0$ and $(D + 1) y = 0$.
$(D - 2) y = 0$ is solved by $y_1 = e^{2x}$, and $(D + 1) y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.

Example 18. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
To match the initial conditions, we need to solve $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$. We find $C_1 = 3$, $C_2 = 1$.
Hence the solution is $y(x) = 3e^{2x} + e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D) y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 19. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. This DE is of the form $p(D)\, y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D + 1)(D + 2)(D + 4)$. (Don't worry! You won't be asked to factor cubic polynomials by hand.)
Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\, e^{-x} + C_2\, e^{-2x} + C_3\, e^{-4x}$.

Armin Straub
straub@southalabama.edu

This approach applies to any homogeneous linear DE with constant coefficients!

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following result provides the missing solutions.

Theorem 20. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j = 0, 1, \ldots, k-1$.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D) = q(D)\,(D-r)^k$.

We need to find k solutions to the simpler DE $(D-r)^k y = 0$.

It is natural to look for solutions of the form $y = c(x)e^{rx}$.

[We know that $c(x) = 1$ provides a solution. Note that this is the same idea as for variation of constants.]

Note that $(D-r)[c(x)e^{rx}] = (c'(x)e^{rx} + c(x)re^{rx}) - rc(x)e^{rx} = c'(x)e^{rx}$.

Repeating, we get $(D-r)^2[c(x)e^{rx}] = (D-r)[c'(x)e^{rx}] = c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}] = c^{(k)}(x)e^{rx}$.

In particular, $(D-r)^k y = 0$ is solved by $y = c(x)e^{rx}$ if and only if $c^{(k)}(x) = 0$.

The DE $c^{(k)}(x) = 0$ is clearly solved by x^j for $j = 0, 1, \ldots, k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 21. Find the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Solution. The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 20, we have the solutions $y(x) = x^j e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Example 22. Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.

By Theorem 20, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Armin Straub
straub@southalabama.edu