Midterm #2 – Practice

MATH 332 — Differential Equations II
Thursday, Nov 5

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Suppose that the matrix A satisfies $e^{Ax} = \frac{1}{7}\begin{bmatrix} e^{-9x} + 6e^{-2x} & -2e^{-9x} + 2e^{-2x} \\ -3e^{-9x} + 3e^{-2x} & 6e^{-9x} + e^{-2x} \end{bmatrix}$.

(a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 3e^{x} \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) What is A?

Problem 2.

(a) Suppose $y(x)$ has the power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. How can we compute the a_n from $y(x)$?

(b) Suppose $f(t)$ has the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$.

How can we compute the a_n and b_n from $f(t)$?

Problem 3. Spell out the power series (around $x = 0$) of the following functions.

(a) e^{-3x}

(b) $\sin(3x^2)$

(c) $\frac{5}{1 + 7x^2}$

Problem 4. Let $y(x)$ be the unique solution to the IVP $y'' = x + 2y^3$, $y(0) = 1$, $y'(0) = 2$.

Determine the first several terms (up to x^4) in the power series of $y(x)$.

Problem 5. Consider the DE $y'' = x(x^2 + 7)\, y' + (x^2 + 3)y$.

Derive a recursive description of a power series solutions $y(x)$.

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Problem 6. Find a minimum value for the radius of convergence of a power series solution to $(4x^2+1)y'' = \frac{y}{x+1}$ at $x=3$.

Problem 7. Derive a recursive description of the power series for $y(x) = \dfrac{1}{1-2x-5x^2}$.

Problem 8. A mass-spring system is described by the equation

$$my'' + y = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

(a) For which m does resonance occur?

(b) Find the general solution when $m = 1/9$.

Problem 9. Consider the function $f(t) = 2(1-t)$, defined for $t \in [0,1]$.

(a) Sketch the Fourier series of $f(t)$ for $t \in [-4,4]$.

(b) Sketch the Fourier cosine series of $f(t)$ for $t \in [-4,4]$.

(c) Sketch the Fourier sine series of $f(t)$ for $t \in [-4,4]$.

In each sketch, carefully mark the values of the Fourier series at discontinuities.

Problem 10. Compute the Fourier sine series of the function $f(t)$, defined for $t \in (0,L)$, with $f(t) = 3$.

Problem 11. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

(Make sure to consider all cases.)

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