Review. The heat equation: $u_t = k u_{xx}$

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $u(x, 0) = f(x)$ (IC)

  This specifies an initial temperature distribution at time $t = 0$.

- **Boundary condition** at $x = 0$ and $x = L$: (BC)

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0, t) = A$, $u(L, t) = B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0, t) = u_x(L, t) = 0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0, t) = A$, $u(L, t) = B$ into $u(0, t) = u(L, t) = 0$ by using the fact that $u(t, x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?

Example 128. (cont'd) To get a feeling, let us find some solutions to $u_t = u_{xx}$.

- $u(x, t) = ax + b$ is a solution.
- For instance, $u(x, t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- More interesting are $u(x, t) = e^{-t}\cos(x)$ and $u(x, t) = e^{-t}\sin(x)$.
- More generally, $e^{-n^2 t}\cos(nx)$ and $e^{-n^2 t}\sin(nx)$ are solutions.

Important observation. This actually reveals a strategy for solving the PDE $u_t = u_{xx}$ with conditions such as:

$$\begin{aligned} &u(0, t) = u(\pi, t) = 0 && \text{(BC)} \\ &u(x, 0) = f(x), \quad x \in (0, L) && \text{(IC)} \end{aligned}$$

Namely, the solutions $u_n(x, t) = e^{-n^2 t}\sin(nx)$ all satisfy (BC).
It remains to satisfy (IC). Note that $u_n(x, 0) = \sin(nx)$. To find $u(x, t)$ such that $u(x, 0) = f(x)$, we can write $f(x)$ as a Fourier sine series (i.e. extend $f(x)$ to a 2π-periodic odd function):

$$f(x) = \sum_{n \geqslant 1} b_n \sin(nx)$$

Then $u(x, t) = \sum_{n \geqslant 1} b_n u_n(x, t) = \sum_{n \geqslant 1} b_n e^{-n^2 t}\sin(nx)$ solves the PDE $u_t = u_{xx}$ with (BC) and (IC).

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Example 129. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} & u_t = k u_{xx} && \text{(PDE)} \\ & u(0,t) = u(L,t) = 0 && \text{(BC)} \\ & u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We will first look for simple solutions of $\text{(PDE)}+\text{(BC)}$ (and then we plan to take a combination of such solutions that satisfies (IC) as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called **separation of variables** and it is crucial for solving other PDEs as well.

- Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$.

  Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$. Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

- Consider (BC). Note that $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$.

  [Because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution.]

  Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.

- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin\left(\frac{\pi n}{L} x\right)$ corresponding to the eigenvalues $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3 \ldots$.

- On the other hand, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right)$ solving $\text{(PDE)}+\text{(BC)}$.

- We wish to combine these in such a way that (IC) holds as well.

  At $t = 0$, $u_n(x,0) = \sin\left(\frac{\pi n}{L} x\right)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0, L)$, to an odd $2L$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right)$.

Consequently, $\text{(PDE)}+\text{(BC)}+\text{(IC)}$ is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, u_n(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right).$$

Example 130. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} & u_t = u_{xx} \\ & u(0,t) = u(1,t) = 0 \\ & u(x,0) = 1, \quad x \in (0,1) \end{aligned}$$

Solution. This is the case $k = 1$, $L = 1$ and $f(x) = 1$, $x \in (0,1)$, of the previous example.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 111. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n \pi x).$$

Hence, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n \pi x)$.

Comment. Note that, for $t > 0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

Make some 3D plots!

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The boundary conditions in the next example model insulated ends.

Example 131. Find the unique solution $u(x,t)$ to:

$$\begin{array}{lr} u_t = k u_{xx} & \text{(PDE)} \\ u_x(0,t) = u_x(L,t) = 0 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,L) & \text{(IC)} \end{array}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).
  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.
- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is left as a homework to show that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{\pi n}{L}x\right)$ corresponding to $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3 \ldots$. [See practice problems.]
- On the other hand (as before), T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L}x\right)$ solving (PDE)+(BC).
- We wish to combine these in such a way that (IC) holds.
  At $t = 0$, $u_n(x,0) = \cos\left(\frac{\pi n}{L}x\right)$. All of these are $2L$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n}{L}x\right)$.
  Note that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \frac{a_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L}x\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

Comment. We indicated earlier that

$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)} \\ &u(0,t) = a, \quad u(L,t) = b && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

can be solved by realizing that $Ax + B$ solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L}x$ (so that $v(0) = a$ and $v(L) = b$). We then look for a solution of the form $u(x,t) = v(x) + w(x,t)$. Note that $u(x,t)$ solves (PDE)+(BC)+(IC) if and only if $w(x,t)$ solves:

$$\begin{aligned} &w_t = k w_{xx} && \text{(PDE)} \\ &w(0,t) = 0, \quad w(L,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

This the (homogeneous) heat equation that we know how to solve.

$v(x)$ is called the **steady-state solution** (it does not depend on time!) and $w(x,t)$ the **transient solution** (note that $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$).

Example 132. Consider the heat flow problem:

$$\begin{aligned} &u_t = 3u_{xx} + 4x^2 && \text{(PDE)} \\ &u(0,t) = 1, \quad u_x(3,t) = -5 && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,3) && \text{(IC)} \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where $w(x,t)$ is the transient solution which (together with its derivatives) tends to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$. Note that this also implies that $w_t = 3w_{xx}$.
- Plugging into (BC), we get $v(0) + w(0,t) = 1$ and $v'(3) + w_x(3,t) = -5$. Letting $t \to \infty$, these become $v(0) = 1$ and $v'(3) = -5$.
- Solving the ODE $0 = 3v'' + 4x^2$ with boundary conditions $v(0) = 1$ and $v'(3) = -5$, we find

  $$v(x) = \iint -\frac{4}{3}x^2 \mathrm{d}x\mathrm{d}x = -\frac{1}{9}x^4 + C_1 + C_2 x$$

  and therefore the steady-state solution $v(x) = -\frac{1}{9}x^4 + 1 + 7x$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 3w_{xx} && \text{(PDE*)} \\ &w(0,t) = 0, \quad w_x(3,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x) && \text{(IC*)} \end{aligned}$$

We know how to solve this homogeneous heat flow problem (see practice problems) using separation of variables.

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Steady-state temperature

Review. (2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III (more below).

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

Comment. Also known as the "potential equation"; satisfied by electric/gravitational potential functions.

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

Other notations. $\Delta f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f = \nabla^2 f$

Boundary conditions. For steady-state temperatures profiles, it is natural to prescribe the temperature on the boundary of a region $R \subseteq \mathbb{R}^2$ (or $R \subseteq \mathbb{R}^3$ in the 3D case).

Comment. Gravitational and electrostatic potentials (not in the vacuum) satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

In our next example we solve the Dirichlet problem in the case when R is a rectangle.

Important observation. We are using homogeneous boundary conditions for three of the sides. That is actually no loss of generality.

Indeed, note that in order to solve
$$\begin{array}{rl} u_{xx} + u_{yy} = 0 & \text{(PDE)} \\ u(x,0) = f_1(x) & \\ u(x,b) = f_2(x) & \\ u(0,y) = f_3(y) & \text{(BC)} \\ u(a,y) = f_4(y) & \end{array}$$
we can solve the four Dirichlet problems:

$$\begin{array}{llll} u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 \\ u(x,0) = f_1(x) & u(x,0) = 0 & u(x,0) = 0 & u(x,0) = 0 \\ u(x,b) = 0 & u(x,b) = f_2(x) & u(x,b) = 0 & u(x,b) = 0 \\ u(0,y) = 0 & u(0,y) = 0 & u(0,y) = f_3(y) & u(0,y) = 0 \\ u(a,y) = 0 & u(a,y) = 0 & u(a,y) = 0 & u(a,y) = f_4(y) \end{array}$$

The sum of the four solutions then solves the Dirichlet problem we started with.

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Example 133. Find the unique solution $u(x,y)$ to:

$$\begin{array}{ll} u_{xx}+u_{yy}=0 & \text{(PDE)} \\ u(x,0) = f(x) & \\ u(x,b) = 0 & \text{(BC)} \\ u(0,y) = 0 & \\ u(a,y) = 0 & \end{array}$$

Solution.

- We proceed as before and look for solutions $u(x,y)=X(x)Y(y)$ (**separation of variables**).

  Plugging into (PDE), we get $X''(x)Y(y)+X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{const}=:-\lambda$.
  We thus have $X''+\lambda X=0$ and $Y''-\lambda Y=0$.

- From the last three (BC), we get $X(0)=0$, $X(a)=0$, $Y(b)=0$.
  We ignore the first (inhomogeneous) condition for now.

- So X solves $X''+\lambda X=0$, $X(0)=0$, $X(a)=0$.
  From earlier, we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x)=\sin\left(\frac{\pi n}{a}x\right)$ corresponding to $\lambda=\left(\frac{\pi n}{a}\right)^2$, $n=1,2,3\ldots$.

- On the other hand, Y solves $Y''-\lambda Y=0$, and hence $Y(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y}$.

  The condition $Y(b)=0$ implies that $Ae^{\sqrt{\lambda}b}+Be^{-\sqrt{\lambda}b}=0$ so that $B=-Ae^{2\sqrt{\lambda}b}$.

  Hence, $Y(y)=A\left(e^{\sqrt{\lambda}y}-e^{-\sqrt{\lambda}(y-2b)}\right)$.

- Taken together, we have the solutions $u_n(x,y)=\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{-\frac{\pi n}{a}(y-2b)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0)=f(x)$.

- We wish to combine these in such a way that $u(x,0)=f(x)$ holds as well.

  At $y=0$, $u_n(x,0)=\sin\left(\frac{\pi n}{a}x\right)(1-e^{2\pi nb/a})$. All of these are $2a$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,a)$, to an odd $2a$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{\pi n}{a}x\right)$.
  Note that

  $$b_n=\frac{1}{a}\int_{-a}^{a}f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC) is solved by

$$u(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}u_n(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{-\frac{\pi n}{a}(y-2b)}\right),$$

where

$$b_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.$$

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Example 134. Find the unique solution $u(x,y)$ to:

$$\begin{array}{rl} u_{xx}+u_{yy}=0 & \text{(PDE)} \\ u(x,0)=1 & \\ u(x,2)=0 & \text{(BC)} \\ u(0,y)=0 & \\ u(1,y)=0 & \end{array}$$

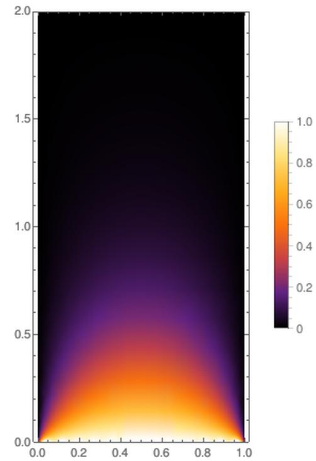
Solution. This is the special case of the previous example with $a=1$, $b=2$ and $f(x)=1$ for $x\in(0,1)$.
From Example 111, we know that $f(x)$ has the Fourier sine series

$$f(x)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi x),\qquad x\in(0,1).$$

Hence,

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{-\pi n(y-4)}).$$

Comment. The temperature at the center is $u(\frac{1}{2},1)\approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).



Example 135. Find the unique solution $u(x,y)$ to:

$$\begin{array}{rl} u_{xx}+u_{yy}=0 & \text{(PDE)} \\ u(x,0)=0 & \\ u(x,2)=3 & \text{(BC)} \\ u(0,y)=0 & \\ u(1,y)=0 & \end{array}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:
Let $v(x,y)=u(x,2-y)$. Then $v_{xx}+v_{yy}=0$, $v(x,0)=3$, $v(x,2)=0$, $v(0,y)=0$, $v(1,y)=0$.
Hence, it follows from the previous example that

$$v(x,y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{-\pi n(y-4)}).$$

Consequently,

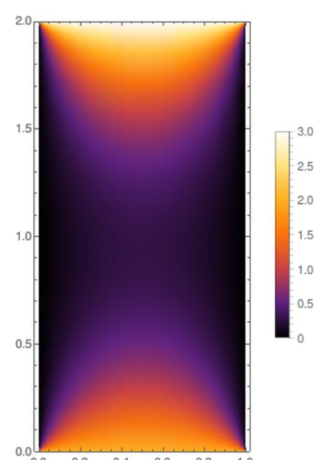
$$u(x,y)=v(x,2-y)=3\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n(2-y)}-e^{\pi n(2+y)}).$$

Example 136. Find the unique solution $u(x,y)$ to:

$$\begin{array}{rl} & u_{xx}+u_{yy} = 0 \\ u(x,0)=2, & u(x,2)=3 \\ u(0,y)=0, & u(1,y)=0 \end{array}$$

Solution. Note that $u(x,y)$ is a combination of the solutions to the previous two examples!

$$u(x,y)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{\sin(\pi n x)}{1-e^{4\pi n}}[2(e^{\pi n y}-e^{-\pi n(y-4)})+3(e^{\pi n(2-y)}-e^{\pi n(2+y)})].$$



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