Review. Theorem 88: If x_0 is an ordinary point of a linear IVP, then it is guaranteed to have a power series solution $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Moreover, its radius of convergence is at least the distance between x_0 and the closest singular point.

Example 92. Find a minimum value for the radius of convergence of a power series solution to $(x^2+4)y''-3xy'+\frac{1}{x+1}y=0$ at $x=2$.

Solution. The singular points are $x=\pm 2i,-1$. Hence, $x=2$ is an ordinary point of the DE and the distance to the nearest singular point is $|2-2i|=\sqrt{2^2+2^2}=\sqrt{8}$ (the distances are $|2-(-1)|=3$, $|2-2i|=|2-(-2i)|=\sqrt{8}$). By Theorem 88, the DE has power series solutions about $x=2$ with radius of convergence at least $\sqrt{8}$.

Example 93. (caution!) Theorem 88 only holds for linear DEs! For nonlinear DEs, it is very hard to predict whether there is a power series solution and what its radius of convergence is.

Consider, for instance, the nonlinear DE $y'+2xy^2=0$.

Its coefficients have no singularities.

A solution to this DE is $y(x)=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (check that!), which has radius of convergence 1.

On the other hand. $y(x)$ also solves the linear DE $(1+x^2)y'+2xy=0$. Note how the DE has singular points for $x=\pm i$. This allows us to predict that $y(x)$ must have a power series with radius of convergence at least 1.

Example 94. (Bessel functions) Consider the DE $x^2y''+xy'+x^2y=0$. Derive a recursive description of a power series solutions $y(x)$ at $x=0$.

Caution! Note that $x=0$ is a singular point (the only) of the DE. Theorem 88 therefore does not guarantee a basis of power series solutions. [However, $x=0$ is what is called a **regular singular point**; for these, we are guaranteed one power series solution, as well as additional solutions expressed using logarithms and power series.]

Comment. We could divide the DE by x (but that wouldn't really change the computations below). The reason for not dividing that x is that this DE is the special case $\alpha=0$ of the **Bessel equation** $x^2y''+xy'+(x^2-\alpha^2)y=0$ (for which no such dividing is possible).

Solution. (plug in power series) Let us spell out power series for x^2y, xy', x^2y'' starting with $y(x)=\sum_{n=0}^{\infty} a_n x^n$:

$$x^2y(x)=\sum_{n=0}^{\infty} a_n x^{n+2}=\sum_{n=2}^{\infty} a_{n-2}x^n$$

$$xy'(x)=\sum_{n=1}^{\infty} n a_n x^n \qquad \left(\text{because } y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}\right)$$

$$x^2y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^n \qquad \left(\text{because } y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right)$$

Hence, the DE becomes $\sum_{n=2}^{\infty} n(n-1)a_n x^n+\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=2}^{\infty} a_{n-2}x^n=0$. We compare coefficients of x^n:

- $n=1$: $a_1=0$
- $n\geqslant 2$: $n(n-1)a_n+na_n+a_{n-2}=0$, which simplifies to $n^2a_n=-a_{n-2}$.

  It follows that $a_{2n}=\frac{(-1)^n}{4^n n!^2}a_0$ and $a_{2n+1}=0$.

Observation. The fact that we found $a_1=0$ reflects the fact that we cannot represent the general solution through power series alone.

Comment. If $a_0=1$, the function we found is a **Bessel function** and denoted as $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!^2}\left(\frac{x}{2}\right)^{2n}$.

The more general Bessel functions $J_\alpha(x)$ are solutions to the DE $x^2y''+xy'+(x^2-\alpha^2)y=0$.

Example 95. (caution!) Consider the linear DE $x^2y' = y - x$. Does it have a convergent power series solution at $x = 0$?

Important note. The DE $x^2y' = y - x$ has the singular point $x = 0$. Hence, Theorem 88 does not apply.

Solution. Let us look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$x^2y'(x) = x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1)a_{n-1}x^n$$

Hence, $x^2y' = y - x$ becomes $\sum_{n=2}^{\infty} (n-1)a_{n-1}x^n = \sum_{n=0}^{\infty} a_n x^n - x$. We compare coefficients of x^n:

- $n = 0$: $a_0 = 0$.
- $n = 1$: $0 = a_1 - 1$, so that $a_1 = 1$.
- $n \geqslant 2$: $(n-1)a_{n-1} = a_n$, from which it follows that $a_n = (n-1)a_{n-1} = (n-1)(n-2)a_{n-2} = \cdots = (n-1)!a_1 = (n-1)!$.

Hence the DE has the "formal" power series solution $y(x) = \sum_{n=1}^{\infty} (n-1)!x^n$.

However, that series is divergent for all $x \neq 0$; that is, the radius of convergence is 0.

Inverses of power series

Example 96. (extra) For each of the following compute the first few terms of the power series.

(a) $(a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 \ldots)$

(b) $\dfrac{1}{a_0 + a_1x + a_2x^2 + \ldots}$

(c) $\dfrac{1}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots}$

Solution.

(a) $a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + O(x^3)$

(b) The answer is $b_0 + b_1x + \ldots$ with the property that $(a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 \ldots) = 1$. By the first part, and comparing coefficients, $a_0b_0 = 1$, $a_0b_1 + a_1b_0 = 0$, $a_0b_2 + a_1b_1 + a_2b_0 = 0$, ... Hence, $b_0 = \frac{1}{a_0}$, $b_1 = -\frac{1}{a_0}(a_1b_0) = -\frac{a_1}{a_0^2}$, $b_2 = -\frac{1}{a_0}(a_1b_1 + a_2b_0) = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}$.

(c) $\dfrac{1}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots} = 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + \ldots$

Comment. This reflects $\frac{1}{e^x} = e^{-x}$.

Likewise, we could compute the first few terms of the power series of, say, $\frac{1}{1-x-x^2}$.

However, it turns out that we can describe all terms in that power series:

Example 97. Derive a recursive description of the power series for $y(x) = \frac{1}{1-x-x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1-x-x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1 = a_0$.
- $n=1$: $0 = a_1 - a_0$, so that $a_1 = a_0 = 1$.
- $n \geqslant 2$: $0 = a_n - a_{n-1} - a_{n-2}$ or, equivalently, $a_n = a_{n-1} + a_{n-2}$.

This is the recursive description of the Fibonacci numbers F_n! In particular $a_n = F_n$.

The first few terms. $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$

Comment. The function $y(x)$ is said to be a **generating function** for the Fibonacci numbers.

Challenge. Can you rederive Binet's formula from partial fractions and the geometric series?

Power series of familiar functions

(Unless we specify otherwise, power series are meant to be about $x=0$.)

Example 98. The **hyperbolic cosine** $\cosh(x)$ is defined to be the even part of e^x. In other words, $\cosh(x)=\frac{1}{2}(e^x+e^{-x})$. Determine its power series.

Solution. It follows from $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ that $\cosh(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Comment. Note that $\cosh(ix)=\cos(x)$ (because $\cos(x)=\frac{1}{2}(e^{ix}+e^{-ix})$).

Comment. The hyperbolic sine $\sinh(x)$ is similarly defined to be the odd part of e^x.

Example 99. (geometric series) Determine $y(x)=\sum_{n=0}^{\infty}x^n$.

Solution. Note that $xy=y-1$. Hence, $y=\frac{1}{1-x}$.

Comment. The radius of convergence of this series is 1. This is easy to see directly. But note that it also follows from Theorem 88 since $y(x)$ solves the "differential" (inhomogeneous) equation $(1-x)y=1$, for which the only singular point is $x=1$.

Example 100. Determine a power series for $\frac{1}{1+x^2}$.

Solution. Replace x with $-x^2$ in $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ to get $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Example 101. (extra) Determine a power series for $\ln(x)$ around $x=1$.

Solution. This is equivalent to finding a power series for $\ln(x+1)$ around $x=0$ (see the final step).

Observe that $\ln(x+1)=\int\frac{\mathrm{d}x}{1+x}+C$ and that $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$.

Integrating, $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+C$. Since $\ln(1)=0$, we conclude that $C=0$.

Finally, $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$ is equivalent to $\ln(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1}$.

Comment. Choosing $x=2$ in $\ln(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x-1)^{n+1}$ results in $\ln(2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$.
The latter is the alternating harmonic sum.
Can you see from here why the harmonic sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges?

Example 102. Determine a power series for $\arctan(x)$.

Solution. Recall that $\arctan(x)=\int\frac{\mathrm{d}x}{1+x^2}+C$. Hence, we need to integrate $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

It follows that $\arctan(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$. Since $\arctan(0)=0$, we conclude that $C=0$.

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Example 103. (error function) Determine a power series for $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$.

Solution. It follows from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ that $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

Integrating, we obtain $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

Example 104. Determine the first several terms (up to x^5) in the power series of $\tan(x)$.

Solution. Observe that $y(x) = \tan(x)$ is the unique solution to the IVP $y' = 1 + y^2$, $y(0) = 0$.

We can therefore proceed to determine the first few power series coefficients as we did earlier.

That is, we plug $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ into the DE. Note that $y(0) = 0$ means $a_0 = 0$.

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$

$1 + y^2 = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + \ldots)^2 = 1 + a_1^2 x^2 + (2a_1 a_2) x^3 + (2a_1 a_3 + a_2^2) x^4 + \ldots$

Comparing coefficients, we find: $a_1 = 1$, $2a_2 = 0$, $3a_3 = a_1^2$, $4a_4 = 2a_1 a_2$, $5a_5 = 2a_1 a_3 + a_2^2$.

Solving for $a_2, a_3, \ldots$, we conclude that $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$

Comment. The fact that $\tan(x)$ is an odd function translates into $a_n = 0$ when n is even. If we had realized that at the beginning, our computation would have been simplified.

Advanced comment. The full power series is $\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}$.

Here, the numbers B_{2n} are (rather mysterious) rational numbers known as **Bernoulli numbers**.

The radius of convergence is $\pi/2$. Note that this is not at all obvious from the DE $y' = 1 + y^2$. This illustrates the fact that nonlinear DEs are much more complicated than linear ones. (There's no analog of Theorem 88.)

Fourier series

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Advertisement. In Linear Algebra II, we will see the following natural way to look at Fourier series: the functions $1, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots$ are orthogonal to each other (for that to make sense, we need to think of functions as vectors and introduce a natural inner product). In fact, they form an orthogonal basis for the space of piecewise smooth functions. In that setting, the formulas for the coefficients a_n and b_n are nothing but the usual projection formulas for orthogonal projection onto a single vector.

Theorem 105. Every* 2π-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity of f, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t.$$

Comment. Another common way to write Fourier series is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

These two ways are equivalent; we can convert between them using Euler's identity $e^{int} = \cos(nt) + i\sin(nt)$.

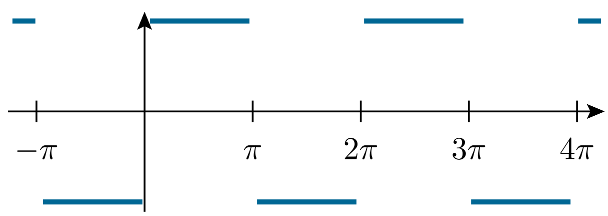
Definition 106. Let $L>0$. $f(t)$ is ***L*-periodic** if $f(t+L)=f(t)$ for all t. The smallest such L is called the **(fundamental) period** of f.

Example 107. The fundamental period of $\cos(nt)$ is $2\pi/n$.

Example 108. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π-periodic for any integer n. And so are their linear combinations. (In other words, 2π-periodic functions form a vector space.)

Example 109. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

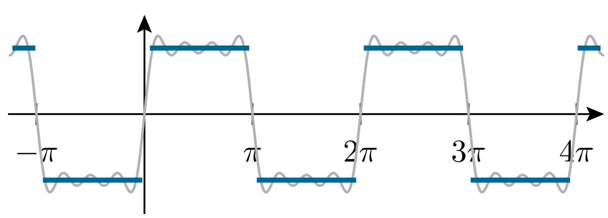
$$f(t)=\begin{cases} -1, & \text{for } t\in(-\pi,0),\\ +1, & \text{for } t\in(0,\pi),\\ 0, & \text{for } t=-\pi,0,\pi. \end{cases}$$



Solution. We compute $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\mathrm{d}t=0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)\mathrm{d}t=\frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t+\int_0^{\pi}\cos(nt)\mathrm{d}t\right]=0\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)\mathrm{d}t=\frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t+\int_0^{\pi}\sin(nt)\mathrm{d}t\right]=\frac{2}{\pi n}[1-\cos(n\pi)]\\ &= \frac{2}{\pi n}[1-(-1)^n]=\begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}. \end{aligned}$$

In conclusion, $f(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(nt)=\frac{4}{\pi}\left(\sin(t)+\frac{1}{3}\sin(3t)+\frac{1}{5}\sin(5t)+\ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t=-\pi,0,\pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: https://en.wikipedia.org/wiki/Gibbs_phenomenon

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Comment. Set $t = \frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n < 10^{-768}$, or $n > 10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not obvious! Recall, for instance, that the (odd part of) the harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges.

There is nothing special about 2π-periodic functions considered last time (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

If $f(t)$ has period $2L$, then $\tilde{f}(x) := f\left(\frac{L}{\pi}t\right)$ has period 2π. Therefore Theorem 105 implies the following:

Theorem 110. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Example 111. Find the Fourier series of the 2-periodic function $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t) = f(\pi t)$, with f as in the previous example, to get $g(t) = f(\pi t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(n\pi t)$.

Theorem 112. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}b_n\cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L}a_n\sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Example 113. Let $h(t)$ be the 2-periodic function with $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n = 0$. Computing a_n is left as an exercise.

Solution. Note that $h(t)$ is continuous and $h'(t) = g(t)$, with $g(t)$ as in Example 111. Hence, we can apply Theorem 112 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(n\pi t) \quad \Longrightarrow \quad h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\left(-\frac{1}{\pi n}\right)\cos(n\pi t) + C,$$

where $C = \frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} h(t)\mathrm{d}t = \frac{1}{2}$ is the constant of integration. Thus, $h(t) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2}\cos(n\pi t)$.

Remark. Note that $t=0$ in the last Fourier series, gives us $\frac{\pi^2}{8}=\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$. As an exercise, you can try to find from here the fact that $\sum_{n\geqslant 1}\frac{1}{n^2}=\frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n\geqslant 1}\frac{1}{n^4}=\frac{\pi^4}{90}$.
Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3),\zeta(5),\ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Example 114. (caution!) The function $g(t)$, from Example 111, is not continuous. For all values, except the discontinuities, we have $g'(t)=0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n \text{ odd}}\cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t=0$). This illustrates that we cannot apply Theorem 112 because of the missing continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

Fourier series and linear differential equations

In the following examples, we consider inhomogeneous linear DEs $p(D)y=F(t)$ where $F(t)$ is a periodic function that can be expressed as a Fourier series. We first review the notion of **resonance** (and how to predict it) and then solve such DEs.

Context. Recall that the inhomogeneous DE $my''+ky=F(t)$ describes, for instance, the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$.

Example 115. Consider the linear DE $my''+ky=\cos(\omega t)$. For which (external) **frequencies** $\omega>0$ does **resonance** occur?

Solution. The roots of $p(D)=mD^2+k$ are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my''+ky=0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0=\sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega=\omega_0$ (overlapping roots).

Review. If $\omega\neq\omega_0$, then there is particular solution of the form $y_p(t)=A\cos(\omega t)+B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t)=A\cos(\omega t)+B\sin(\omega t)+C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega=\omega_0$, then general solution is $y(t)=(C_1+At)\cos(\omega_0 t)+(C_2+Bt)\sin(\omega_0 t)$ and this function is unbounded.

Comment. The inhomogeneous equation $my''+ky=F(t)$ describes the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$.

Example 116. A mass-spring system is described by the DE $2y''+32y=\sum_{n=1}^{\infty}\frac{\cos(n\omega t)}{n^2+1}$.

For which ω does resonance occur?

Solution. The roots of $p(D)=2D^2+32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n\in\{1,2,3,\ldots\}$. Equivalently, resonance occurs if $\omega=4/n$ for some $n\in\{1,2,3,\ldots\}$.

Example 117. A mass-spring system is described by the DE $my''+y=\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D)=mD^2+1$ are $\pm i/\sqrt{m}$, so that that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m}=n/3$ for some $n\in\{1,2,3,\ldots\}$. Equivalently, resonance occurs if $m=9/n^2$ for some $n\in\{1,2,3,\ldots\}$.

Though it requires some effort, we already know how to solve $p(D)y = F(t)$ for periodic forces $F(t)$, once we have a Fourier series for $F(t)$. The same approach works for linear equations of higher order, or even systems of equations.

Example 118. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10\sum_{n \text{ odd}} \frac{4}{\pi n}\sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2 y_p$ (why?!), we get

$$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \overset{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.

- We combine the particular solutions found in the previous step, to see that

$$2y'' + 32y = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(\pi n t) \quad \text{is solved by} \quad y_p = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Example 119. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$. [Exercise!]
- We next solve the equation $2y'' + 32y = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that **resonance** occurs for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $y_p(t) = \frac{\sin(nt)}{32 - 2n^2}$. [Note how this fails for $n = 4$!]

  For $2y'' + 32y = \sin(4t)$, we begin with $y_p = At\cos(4t) + Bt\sin(4t)$. Then $y_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $y_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p'' + 32y_p = 16B\cos(4t) - 16A\sin(4t) \overset{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $y_p = -\frac{1}{16}t\cos(4t)$.

- We combine the particular solutions to get that our DE

$$2y'' + 32y = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$$

  is solved by

$$y_p(t) = \frac{5}{16}t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32 - 2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t=0$ and $t=L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n t}{L}\right) + b_n \sin\left(\frac{2\pi n t}{L}\right)\right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos\left(\frac{\pi n t}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{\pi n t}{L}\right)$.

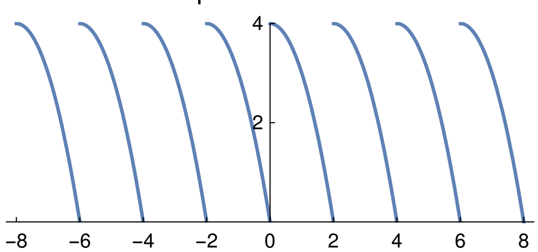
Example 120. Consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

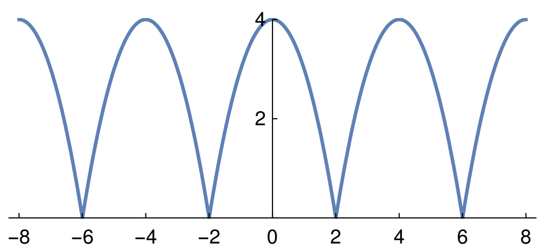
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

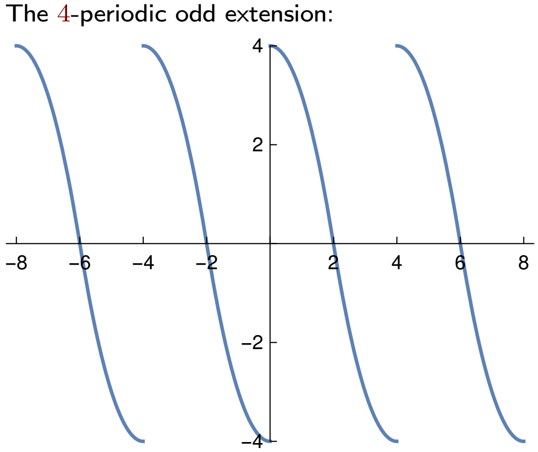
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



Example 121. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

(a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.

(b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.

(c) Let $H(t)$ be the Fourier cosine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

(a) Note that the extension of $f(t)$ has discontinuities at $\ldots, -2, 0, 2, 4, \ldots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:
$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$
$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$
$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

(b) $G(2) = f(2) = 0$ (see plot!)
[note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous]
$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$
$G\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

(c) $H(2) = \frac{1}{2}(f(2^-) - f(2^-)) = 0$ (see plot!)
[note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps]
$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$
$H\left(-\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$

Boundary value problems

Example 122. The **IVP** (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$.

Initial value problems are often used when the problem depends on time. Then, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next example illustrates that such a boundary value problem (BVP) may or may not have a unique solution.

Example 123. Verify the following claims.

(a) The BVP $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

(b) The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

Solution.

(a) We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well.

(b) This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition is true for any B.

It is therefore natural to ask: for which λ does the BVP $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ have nonzero solutions? (We assume that $L > 0$.)

Such solutions are called **eigenfunctions** and λ is the corresponding **eigenvalue**.

Remark. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue.

Example 124. Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.

Such a problem is called an **eigenvalue problem**.

Solution. The solutions of the DE look different in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, so we consider them individually.

$\boldsymbol{\lambda = 0}$. Then $y(x) = Ax + B$ and $y(0) = y(L) = 0$ implies that $y(x) = 0$. No eigenfunction here.

$\boldsymbol{\lambda < 0}$. The roots of the characteristic polynomial are $\pm\sqrt{-\lambda}$. Writing $\rho = \sqrt{-\lambda}$, the general solution therefore is $y(x) = Ae^{\rho x} + Be^{-\rho x}$. $y(0) = A + B \overset{!}{=} 0$ implies $B = -A$. Using that, we get $y(L) = A(e^{\rho L} - e^{-\rho L}) \overset{!}{=} 0$. For eigenfunctions we need $A \neq 0$, so $e^{\rho L} = e^{-\rho L}$ which implies $\rho L = -\rho L$. This cannot happen since $\rho \neq 0$ and $L \neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda > 0}$. The roots of the characteristic polynomial are $\pm i\sqrt{\lambda}$. Writing $\rho = \sqrt{\lambda}$, the general solution thus is $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using that, $y(L) = B\sin(\rho L) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(\rho L) = 0$. This happens if $\rho L = n\pi$ for $n = 1, 2, 3, \ldots$ (since ρ and L are both positive). Equivalently, $\rho = \frac{n\pi}{L}$. Consequently, we find the eigenfunctions $y_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$, with eigenvalue $\lambda = \left(\frac{n\pi}{L}\right)^2$.

Example 125. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0, L]$. The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy'' + Py = 0$, $y(0) = 0$, $y(L) = 0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$, with $\lambda = \frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda = \left(\frac{\pi n}{L}\right)^2$ for some $n = 1, 2, 3, \ldots$, means that buckling can only occur if $P = \left(\frac{\pi n}{L}\right)^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

https://en.wikipedia.org/wiki/Euler%27s_critical_load

Partial differential equations

The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x,t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0, L]$.

Notation. $u(x,t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t} u$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = k u_{xx}$, with $k > 0$.

(heat equation)

$$u_t = k u_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1 u_1 + c_2 u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial/\partial x, \partial/\partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

Example 126. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 127. To get a feeling, let us find some other solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- ... to be continued ...
  Can you find further solutions?