A crash course in linear algebra

Example 1. A typical 2×3 matrix is $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$.

It is composed of column vectors like $\begin{bmatrix}2\\5\end{bmatrix}$ and row vectors like $[\,1\;\;2\;\;3\,]$.

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar:

For instance, $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}+\begin{bmatrix}1&0&2\\2&3&-1\end{bmatrix}=\begin{bmatrix}2&2&5\\6&8&5\end{bmatrix}$ or $3\cdot\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}3&6&9\\12&15&18\end{bmatrix}$.

Remark. More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, $\ldots$

Example 2. The **transpose** A^T of A is obtained by interchanging roles of rows and columns.

For instance. $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}^T=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$

Example 3. Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication $[\,a\;\;b\;\;c\,]\begin{bmatrix}x\\y\\z\end{bmatrix}=ax+by+cz$ of row and column vectors.

For instance. $\begin{bmatrix}1&-1&1\\2&1&3\end{bmatrix}\begin{bmatrix}1&0\\-1&1\\2&-2\end{bmatrix}=\begin{bmatrix}4&-3\\7&-5\end{bmatrix}$

In general, we can multiply a $m\times n$ matrix A with a $n\times r$ matrix B to get a $m\times r$ matrix AB.

Its entry in row i and column j is defined to be $(AB)_{ij}=(\text{row } i \text{ of } A)\begin{bmatrix}\text{column}\\ j\\ \text{of } B\end{bmatrix}$.

Comment. One way to think about the multiplication $A\boldsymbol{x}$ is that the resulting vector is a linear combination of the columns of A with coefficients from $\boldsymbol{x}$. Similarly, we can think of $\boldsymbol{x}^TA$ as a combination of the rows of A.

Some nice properties of matrix multiplication are:

- There is an $n\times n$ identity matrix I (all entries are zero except the diagonal ones which are 1). It satisfies $AI=A$ and $IA=A$.
- The associative law $A(BC)=(AB)C$ holds. Hence, we can write ABC without ambiguity.
- The distributive laws including $A(B+C)=AB+AC$ hold.

Example 4. $\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}\neq\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}$, so we have no commutative law.

Example 5. $\begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{bmatrix}1&-1\\-2&3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Hence, the two matrices on the left are inverses of each other: $\begin{bmatrix}3&1\\2&1\end{bmatrix}^{-1}=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}$, $\begin{bmatrix}1&-1\\-2&3\end{bmatrix}^{-1}=\begin{bmatrix}3&1\\2&1\end{bmatrix}$.

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The **inverse** A^{-1} of a matrix A is characterized by $A^{-1}A = I$ and $AA^{-1} = I$.

Example 6. The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad-bc \neq 0$$

Let's check that! $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = I_2$

In particular, a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Longleftrightarrow$ $ad-bc \neq 0$.

Recall that this is the **determinant**: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$.

$$\det(A) = 0 \quad \Longleftrightarrow \quad A \text{ is not invertible}$$

Example 7. The system $\begin{array}{rl} 7x_1 - 2x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{array}$ is equivalent to $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Solve it.

Solution. Multiplying (from the left!) by $\begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$ produces $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which gives the solution of the original equations.

The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 \quad &\Longleftrightarrow \quad A \text{ is invertible} \\ &\Longleftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ for all } \boldsymbol{b} \\ &\Longleftrightarrow \quad A\boldsymbol{x} = 0 \text{ is only solved by } \boldsymbol{x} = 0 \end{aligned}$$

Example 8. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$, which appeared in the formula for the inverse.

Example 9. (extra) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

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Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equation (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 10. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 11. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 12. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \to \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.

Excursion: Euler's identity

Theorem 13. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

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Example 14. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.

Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$ (that's what we actually did in class).

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.

These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

Review: Linear DEs

The most general first-order linear DE is $y' = a(x)y + f(x)$.

We will recall next time that we can always solve it.

The corresponding **homogeneous** linear DE is $y' = a(x)y$.

Important comment. Write $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we can write $y' - a(x)y = f(x)$ as $Ly = f(x)$ where $L = D - a(x)$.

The corresponding homogeneous DE is simply $Ly = 0$.

More next time!

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Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Linear first-order DEs

The following DE is linear and first-order (but not with constant coefficients).

Example 15. Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + C$, so that $|y| = e^{\frac{1}{3}x^2 + C}$. Since the RHS is never zero, $y = \pm e^C e^{\frac{1}{3}x^3} = D e^{\frac{1}{3}x^3}$ (with $D = \pm e^C$). Note that $D = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = D e^{\frac{1}{3}x^3}$ (with D any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 16. Solve $y' = a(x) y$.

Solution. Proceeding as in the previous example, we find $y(x) = D e^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.

Recall that, to find the general solution of the inhomogeneous DE $y' = a(x) y + f(x)$, we only need to find a particular solution y_p.

Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE $y' = a(x) y$.

Theorem 17. (variation of constants) $y' = a(x) y + f(x)$ has the particular solution

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} \mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x) y$.

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Proof. $y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y + f(x)$ □

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x)y_h(x)$.

If we plug $y(x) = c(x)y_h(x)$ into the DE, we find $c'y_h + cy_h' = acy_h + f$. Since $y_h' = ay_h$, this simplifies to $c'y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$. Hence, $c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x$, which is the formula in the theorem.

Example 18. Solve $x^2y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x)\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$?!) Hence:

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x = \frac{1}{x}\int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

Homogeneous linear DEs with constant coefficients

Example 19. Find the general solution to $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.

Plugging e^{rx} into the DE, we get $r^2e^{rx} - re^{rx} - 2e^{rx} = 0$.

Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.

This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.

Since this a homogeneous linear DE, the general solution is $y = C_1e^{2x} + C_2e^{-x}$.

Next time. A useful way to look at these kinds of differential equations through differential operators.

Homogeneous linear DEs with constant coefficients

Example 20. Find the general solution to $y'' - y' - 2y = 0$.

Solution. (again) We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operators) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.
Note that $D^2 - D - 2 = (D - 2)(D + 1)$ is the **characteristic polynomial**.
It follows that we get solutions to $(D - 2)(D + 1)y = 0$ from $(D - 2)y = 0$ and $(D + 1)y = 0$.
$(D - 2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D + 1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.

Example 21. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
To match the initial conditions, we need to solve $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$. We find $C_1 = 3$, $C_2 = 1$.
Hence the solution is $y(x) = 3e^{2x} + e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 22. Find the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

Solution. This DE is of the form $p(D)\, y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D + 1)(D + 2)(D + 4)$.
Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. That's enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\, e^{-x} + C_2\, e^{-2x} + C_3\, e^{-4x}$.

Example 23. Find the general solution of $y'' + y = 0$.

Solution. The characteristic equation is $r^2 + 1 = 0$ which has no solutions over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1\, e^{ix} + C_2\, e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity

$$e^{ix} = \cos(x) + i \sin(x).$$

Note that $e^{-ix} = \cos(x) - i \sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

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Example 24. Find the general solution of $y'' - 4y' + 13y = 0$.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

Hence, the general solution is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.

Note. $e^{(2+3i)x} = e^{2x}e^{3ix} = e^{2x}(\cos(3x) + i\sin(3x))$

This approach applies to any homogeneous linear DE with constant coefficients!

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following result provides the missing solutions.

Theorem 25. Consider the homogeneous linear DE with constant coefficients $p(D)y = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r = a \pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j = 0, 1, \ldots, k-1$.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D) = q(D)\,(D - r)^k$.

We need to find k solutions to the simpler DE $(D - r)^k y = 0$.

It is natural to look for solutions of the form $y = c(x)e^{rx}$.

[We know that $c(x) = 1$ provides a solution. Note that this is the same idea as for variation of constants.]

Note that $(D - r)[c(x)e^{rx}] = (c'(x)e^{rx} + c(x)re^{rx}) - rc(x)e^{rx} = c'(x)e^{rx}$.

Repeating, we get $(D - r)^2[c(x)e^{rx}] = (D - r)[c'(x)e^{rx}] = c''(x)e^{rx}$ and, eventually, $(D - r)^k[c(x)e^{rx}] = c^{(k)}(x)e^{rx}$.

In particular, $(D - r)^k y = 0$ is solved by $y = c(x)e^{rx}$ if and only if $c^{(k)}(x) = 0$.

The DE $c^{(k)}(x) = 0$ is clearly solved by x^j for $j = 0, 1, \ldots, k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 26. Find the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Solution. The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 25, we have the solutions $y(x) = x^j e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2 x + C_3 x^2$.

Example 27. Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D - 1)^2(D + 2)$ has roots $1, 1, -2$.

By Theorem 25, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Inhomogeneous linear DEs with constant coefficients

Example 28. Find the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. We already know one solution $y_p = 3x$.

The homogeneous DE $p(D)y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]

Therefore, the general solution to the original DE is $y_p + C_1 y_1 + C_2 y_2 = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Next time. How to find the particular solution $y_p = 3x$ ourselves.