Fourier series example

Fourier series

Every* 2L-periodic $f(t)$ can be written as:

$$f(t) \overset{*}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

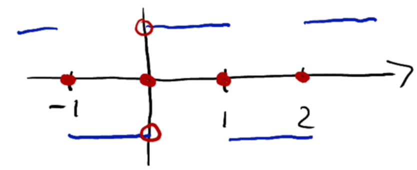
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

EG

$$f(t) = \begin{cases} -1 & \text{for } t \in (-1,0) \\ +1 & \text{for } t \in (0,1) \end{cases}$$

extended 2-periodically

$L = 1$



$$f(0) = \frac{-1+1}{2} = 0$$

value of Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\pi t) + b_n \sin(n\pi t)\right]$$

HW: $a_0 = 0$

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \int_{-1}^{0} \underbrace{f(t)}_{=-1}\cos(n\pi t)\,dt + \int_{0}^{1} \underbrace{f(t)}_{=1}\cos(n\pi t)\,dt$$

$$= -\left[\frac{1}{n\pi}\sin(n\pi t)\right]_{-1}^{0}$$

$\sin(-n\pi) = 0$

$$+ \left[\frac{1}{n\pi}\sin(n\pi t)\right]_{0}^{1}$$

$$= -[0-0] + [0-0] = 0$$

because $f(t)$ is odd

$$\int \cos(n\pi t)\,dt = \frac{1}{n\pi}\sin(n\pi t) + C$$

$$\int_a^b f(t)\,dt = \left[F(t)\right]_a^b = F(b) - F(a)$$

antiderivative

$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)\,dt = -\int_{-1}^{0}\sin(n\pi t)\,dt + \int_{0}^{1}\sin(n\pi t)\,dt$$

$$= \cancel{-}\left[\frac{\cancel{-}1}{n\pi}\cos(n\pi t)\right]_{-1}^{0} + \left[\frac{\cancel{-}1}{n\pi}\cos(n\pi t)\right]_{0}^{1}$$

$\cos(0) = 1$

$$= \left[\cancel{\frac{1}{n\pi}} - \frac{1}{n\pi}\cos(\cancel{-}n\pi)\right] - \left[\frac{1}{n\pi}\cos(n\pi) - \cancel{\frac{1}{n\pi}}\right]$$

$$= \frac{1}{n\pi}\left[2 - 2\underbrace{\cos(n\pi)}_{=(-1)^n}\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$\Longrightarrow$ overall

$$f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin(n\pi t)$$