Fourier series

DEF $f(t)$ is L-periodic if $f(t+L) = f(t)$ for all t.

The smallest such $L>0$ is the fundamental period.

EG fundamental period of $\cos(t)$ is 2π

— " — $\cos(Lt)$ is $\frac{2\pi}{L}$

$L \to \frac{n\pi}{L}$

$\cos\left(\frac{n\pi t}{L}\right)$ is $\frac{2L}{n}$

is $2L$-periodic [n on integers]

EG: piecewise smooth

THM Every* $2L$-periodic $f(t)$ can be written as:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

Fourier series of $f(t)$

$\frac{f(t^-)+f(t^+)}{2}$ if f is not continuous at t

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt$$

Fourier coefficients

Fourier series are a bit finicky to work with:

- power series can be differentiated term-wise

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

- Fourier series, too, but only if continuous

$$\frac{d}{dt}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right)$$