Power series + linear DEs

radius of convergence

review

$\sum_{n=0}^{\infty} a_n (x-x_0)^n$ has a radius of convergence R

- series converges if $|x-x_0| < R$
- series diverges if $|x-x_0| > R$

THM

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \dots + p_1(x) y' + p_0(x) y = f(x)$$

$$y(x_0) = \dots \quad y'(x_0) = \dots \quad \dots \quad y^{(n-1)}(x_0) = \dots$$

If x_0 is an ordinary point, then this IVP has a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{with } R \geq \text{distance from } x_0 \text{ to closest singular point}$$

i.e. not an ordinary point

meaning that $p_0(x), p_1(x), \dots$ as well as $f(x)$ are analytic at $x = x_0$

EG

$$(x^2+1) y''' = \frac{y}{x-5} \qquad x_0 = 2 \quad \text{ordinary!}$$

$$y''' - \frac{1}{(x^2+1)(x-5)} y = 0$$

singular points : $\pm i, 5$

distance from 2 to closest singular point $= \sqrt{5}$

$$|2 \mp i| = \sqrt{2^2 + 1^2} = \sqrt{5} < 3$$

$$|2-5| = 3$$

$|a+bi| = \sqrt{a^2+b^2}$

$\Rightarrow$ DE has power series solutions around $x=2$ with radius of convergence at least $\sqrt{5}$

EG

$$y'' = xy \qquad y(0) = a \qquad y'(0) = b$$

Airy equation

earlier! $y = \sum_{n=0}^{\infty} a_n x^n$

no singular points

$\Rightarrow$ radius of convergence $= \infty$

$y' = y$
$y(0) = 1$
$y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$R = \infty$