

Power series: first examples

review

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad a_n = \frac{y^{(n)}(x_0)}{n!}$$

if $y(x)$ is analytic around $x=x_0$

EG $y(x) = e^x$ $y^{(n)} = e^x$ $y' = y, y(0) = 1$

$$a_n = \frac{y^{(n)}(0)}{n!} = \frac{1}{n!}$$

$\Rightarrow x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \quad 0! = 1$$

EG $e^{7x} = \sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7^n}{n!} x^n$ $a_n = \frac{7^n}{n!}$

EG $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ $y'' = -y, y(0) = 1, y'(0) = 0$

Euler's formula: $e^{ix} = \cos(x) + i \sin(x)$

$\begin{cases} 2 = -1 \\ 3 = -i \\ 4 = 1 \\ \vdots \end{cases}$

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1}$$

$i^{2n} = (-1)^n \cos(x)$

EG $y'' = xy$ $y(0) = a, y'(0) = b$
 Airy equation $Ai(x)$

First terms in power series (around $x_0 = 0$):

$$y = \frac{y(0)}{1} + \frac{y'(0)}{1} x + \frac{y''(0)}{2} x^2 + \frac{y'''(0)}{3!} x^3 + \dots$$

$$y''(0) = 0 \cdot y(0) = 0$$

$$y'' = xy$$

$$y'''(0) = y(0) + 0 \cdot y'(0) = a$$

$$y''' = y + xy'$$

$$y^{(4)}(0) = 2y'(0) + 0 \cdot y''(0) = 2b$$

$$y^{(4)} = y' + y' + xy'' = 2y' + xy''$$

$$\Rightarrow y = a + bx + \frac{a}{6} x^3 + \frac{b}{12} x^4 + \dots$$

HW: more!