Power series

DEF $y(x)$ is analytic around $x = x_0$

$$\iff \quad y(x) = \boxed{\sum_{n=0}^{\infty} a_n (x-x_0)^n} \quad \text{power series}$$

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

THM If $y(x)$ analytic, $\quad a_n = \dfrac{y^{(n)}(x_0)}{n!}$. Taylor series! (Calculus 2)

in particular: infinitely differentiable

$|x|$ not analytic!

caution $y(x) = e^{-1/x^2}$ is infinitely differentiable everywhere but not analytic at $x=0$ $\qquad y^{(n)}(0) = 0$

good news power series behave like polynomials:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

$$y' = \quad a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots$$

- $$y'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$ power series again

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$$

$n \to n+1$ $\qquad n+1=1$

$$\int (x-x_0)^n\,dx = \frac{1}{n+1}(x-x_0)^{n+1} + C$$

- $$\int y(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} + C$$

$$= \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-x_0)^n + C$$

$n \to n-1$ $\qquad n-1=0$

Why $a_n = \dfrac{y^{(n)}(x_0)}{n!}$?

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3$$

$$y'(x) = \quad a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots$$

$y'(x_0) = a_1$

$$y''(x) = \quad 2a_2 + 2\cdot 3\, a_3(x-x_0) + \dots$$

$2\cdot 3 = 3!$

$y''(x_0) = 2a_2$

$\vdots$