# Solving systems of recurrences

$$a_{n+1} = M a_n$$

If $v$ is a $\lambda$-eigenvector of $M$, then $a_n = v \cdot \lambda^n$ is a solution.

$Mv = \lambda v$

$$M a_n = \underbrace{Mv}_{=\lambda v} \cdot \lambda^n = v \cdot \lambda^{n+1} = a_{n+1}$$

**EG** $a_{n+1} = \underbrace{\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}}_{M} a_n$

earlier:

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 3-eigenvector

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $-2$-eigenvector

solutions:

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot 3^n$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot (-2)^n$

general solution: $c_1 \begin{bmatrix} 2 \cdot 3^n \\ 3^n \end{bmatrix} + c_2 \begin{bmatrix} (-2)^n \\ (-2)^n \end{bmatrix}$

$$\begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

fundamental matrix solution $\Phi_n$

(columns are solutions!)

$$\Phi_{n+1} = M \, \Phi_n$$

in particular:

$$\Phi_n = M^n \Phi_0 \quad (\Phi_0^{-1})$$

$$\Rightarrow M^n = \Phi_n \Phi_0^{-1}$$

**EG**

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}^n = \Phi_n \Phi_0^{-1}$$

$$= \begin{bmatrix} 2 \cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3^n - (-2)^n & -2 \cdot 3^n + 2 \cdot (-2)^n \\ 3^n - (-2)^n & -3^n + 2 \cdot (-2)^n \end{bmatrix}$$

check:

$n=1$ ✓ $\begin{bmatrix} 6+2 & -6-4 \\ 3+2 & -3-4 \end{bmatrix}$ =

$n=0$ ✓ $\begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = I$