

# Crash course: eigenvalues + eigenvectors

$A$   $n \times n$

$$Ax = \lambda x \quad \text{with } x \neq 0$$

$\Rightarrow x$  is eigenvector with eigenvalue  $\lambda$

key observation

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

typically only has solution  $x=0$

unless  $\det(A - \lambda I) = 0$

recipe to find eigenvalues + eigenvectors of  $A$

① solve  $\det(A - \lambda I) = 0$

degree  $n$  polynomial: char poly of  $A$

$\Rightarrow$  eigenvalues  $\lambda$

② for each  $\lambda$ , find corresponding eigenvectors  $x$  by solving  $(A - \lambda I)x = 0$

EG  $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

① char. poly:  $|A - \lambda I| = \begin{vmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{vmatrix}$   
 $= (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$

eigenvalues:  $\lambda = 3, -2$

②  $\lambda = 3$ :  $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} x = \vec{0}$   $5x_1 - 10x_2 = 0$

$\Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  or any multiple

$\lambda = -2$ :  $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} x = \vec{0}$   $10x_1 - 10x_2 = 0$   
 $5x_1 - 5x_2 = 0$

$\Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  or any multiple

check

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is 3-eigenvector

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 8 + 1 \cdot (-10) \\ 2 \cdot 5 + 1 \cdot (-7) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is not an eigenvector

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$