Recurrence equations

EG $y' = 7y$

$(D-7)y = 0$

$\Rightarrow \quad y = Ce^{7x}$

EG $y'' - y' - 6y = 0$

$\underbrace{(D^2 - D - 6)}_{(D-3)(D+2)} y = 0$

$\Rightarrow \quad y = C_1 e^{3x} + C_2 e^{-2x}$

EG $a_{n+2} = a_{n+1} + a_n \qquad a_0 = 0 \quad a_1 = 1$

recurrence equation

$n=0: \quad a_2 = a_1 + a_0 = 1 + 0 = 1$

$n=1: \quad a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = 2+1 = 3 \qquad a_5 = 3+2 = 5 \qquad a_6 = 5+3 = 8$

Fibonacci numbers

shift operator
$Na_n = a_{n+1}$

EG $a_{n+1} = 7a_n$

$(N-7)a_n = 0$

$\Rightarrow \quad a_n = C \cdot 7^n$

EG $a_{n+2} - a_{n+1} - 6a_n = 0$

$\underbrace{(N^2 - N - 6)}_{(N-3)(N+2)} a_n = 0$

$\Rightarrow \quad a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$

general solution

EG $a_{n+2} - a_{n+1} - 6a_n = 0 \qquad a_0 = 1 \quad a_1 = 8$

general solution: $a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$

for initial conditions: $\boxed{\begin{aligned} a_0 &= C_1 + C_2 \overset{!}{=} 1 \\ a_1 &= 3C_1 - 2C_2 \overset{!}{=} 8 \end{aligned}}$

$\xrightarrow{\text{solve}} \quad C_1 = 2 \qquad C_2 = -1$

$\Rightarrow \quad a_n = 2 \cdot 3^n - (-2)^n$

note $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ because $|3| > |-2|$

[so that 3^n dominates $(-2)^n$]

$\rightsquigarrow a_n \approx 2 \cdot 3^n$

$\rightsquigarrow \frac{a_{n+1}}{a_n} \approx \frac{\cancel{2} \cdot 3^{n+1}}{\cancel{2} \cdot 3^n} = 3$

Fibonacci numbers

$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$

$\frac{1+\sqrt{5}}{2} \approx 1.618$ golden ratio $\qquad \frac{1-\sqrt{5}}{2} \approx -0.618$

Binet formula

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$

$\frac{5}{3} \approx 1.67 \qquad \frac{8}{5} = 1.6 \qquad \frac{13}{8} = 1.625$