Midterm #2

MATH 332 — Differential Equations II
Tuesday, Nov 12

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 32 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (5 points) Derive a recursive description of a power series solution $y(x)$ of the DE $y'' = (3x^2 - 2)y$.

Solution. Let us spell out the power series for $y, x^2 y, y''$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 3\sum_{n=2}^{\infty} a_{n-2} x^n - 2\sum_{n=0}^{\infty} a_n x^n.$$

We compare coefficients of x^n:

- $n=0$: $2a_2 = -2a_0$, so that $a_2 = -a_0$.
- $n=1$: $6a_3 = -2a_1$, so that $a_3 = -\frac{1}{3} a_1$.
- $n \geqslant 2$: $(n+2)(n+1) a_{n+2} = 3a_{n-2} - 2a_n$
  Equivalently, for $n \geqslant 4$, $a_n = -\frac{2}{n(n-1)} a_{n-2} + \frac{3}{n(n-1)} a_{n-4}$.

In conclusion, the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_2 = -a_0, \quad a_3 = -\frac{1}{3} a_1, \quad a_n = -\frac{2}{n(n-1)} a_{n-2} + \frac{3}{n(n-1)} a_{n-4} \quad \text{for } n \geqslant 4.$$

(The values a_0 and a_1 are the initial conditions.) □

Problem 2. (3 points) Derive a recursive description of the power series for $y(x) = \dfrac{1}{1 - 3x + 2x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1 - 3x + 2x^2) \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=0}^{\infty} a_n x^{n+1} + 2\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $1 = a_0$.
- $n=1$: $0 = a_1 - 3a_0$, so that $a_1 = 3a_0 = 3$.
- $n \geqslant 2$: $0 = a_n - 3a_{n-1} + 2a_{n-2}$ or, equivalently, $a_n = 3a_{n-1} - 2a_{n-2}$.

In conclusion, the power series $\dfrac{1}{1 - 3x + 2x^2} = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_0 = 1, \quad a_1 = 3, \quad a_n = 3a_{n-1} - 2a_{n-2} \quad \text{for } n \geqslant 2.$$ □

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Problem 3. (4 points) Consider the function $f(t)=t$, defined for $t\in[0,1]$.

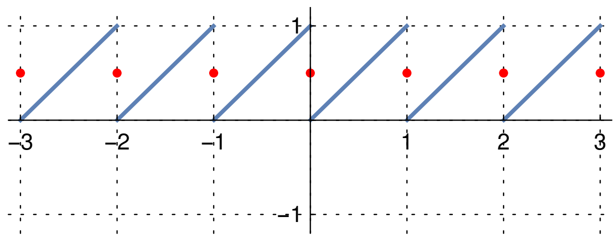
(a) Sketch the Fourier series of $f(t)$ for $t\in[-3,3]$.

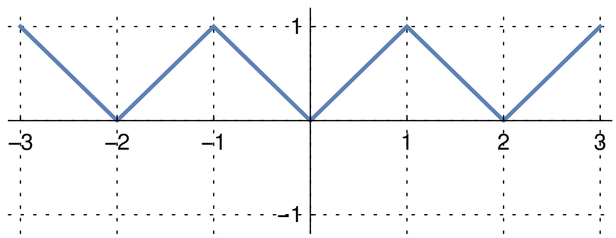
(b) Sketch the Fourier cosine series of $f(t)$ for $t\in[-3,3]$.

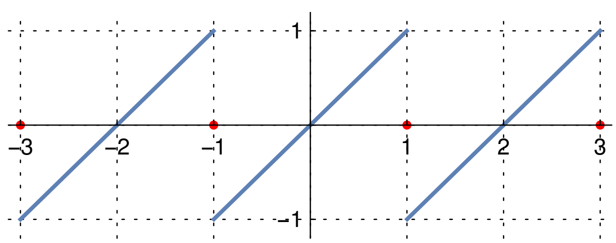
(c) Sketch the Fourier sine series of $f(t)$ for $t\in[-3,3]$.

In each sketch, carefully mark the values of the Fourier series at discontinuities.

Solution.

(a)


(b)


(c)


□

Problem 4. (5 points)

(a) Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$. How can we compute the a_n from $y(x)$? $a_n=$ $\boxed{}$.

(b) Suppose $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{3}\right)+b_n\sin\left(\frac{n\pi t}{3}\right)\right)$. How can we compute the a_n and b_n from $f(t)$?

$a_n=$ $\boxed{}$ and $b_n=$ $\boxed{}$.

(c) Determine the power series around $x = 0$: $e^{7x} = \boxed{}$

(d) Determine the power series around $x = 0$: $\frac{1}{1 + x^2} = \boxed{}$

Solution.

(a) $a_n = \frac{y^{(n)}(x_0)}{n!}$

(b) The Fourier coefficients a_n, b_n can be computed as

$$a_n = \frac{1}{3} \int_{-3}^{3} f(t) \cos\left(\frac{n\pi t}{3}\right) \mathrm{d}t, \qquad b_n = \frac{1}{3} \int_{-3}^{3} f(t) \sin\left(\frac{n\pi t}{3}\right) \mathrm{d}t.$$

(c) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $e^{7x} = \sum_{n=0}^{\infty} \frac{7^n x^n}{n!}$.

(d) Since $\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. □

Problem 5. (3 points) A mass-spring system is described by the equation $y'' + ky = \sum_{n=1}^{\infty} \frac{1}{n^2+7} \cos\left(\frac{nt}{4}\right)$.

For which values of k does resonance occur?

Solution. The roots of $p(D) = D^2 + k$ are $\pm i\sqrt{k}$, so that that the natural frequency is $\sqrt{k}$. Resonance therefore occurs if $\sqrt{k} = n/4$ for some $n \in \{1, 2, 3, \ldots\}$. Equivalently, resonance occurs if $k = n^2/16$ for some $n \in \{1, 2, 3, \ldots\}$. □

Problem 6. (3 points) Find a minimum value for the radius of convergence of a power series solution to

$$(x^2+1)y'' = \frac{y}{x+1} \quad \text{at } x = 2.$$

Solution. Rewriting the DE as $y'' - \frac{1}{(x+1)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, -1$.

Note that $x = 2$ is an ordinary point of the DE and that the distance to the nearest singular point is $|2 - (\pm i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$ (the distance to -1 is $|2 - (-1)| = 3 = \sqrt{9} > \sqrt{5}$).

Hence, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{5}$. □

Problem 7. (2 points) Suppose that the matrix A satisfies $e^{Ax} = \begin{bmatrix} 3e^x - 2e^{2x} & -6e^x + 6e^{2x} \\ e^x - e^{2x} & -2e^x + 3e^{2x} \end{bmatrix}$.

(a) $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is solved by $\boldsymbol{y}(x) = \boxed{}$.

(b) $A = \boxed{}$

Solution.

(a) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^x + 4e^{2x} \\ -e^x + 2e^{2x} \end{bmatrix}$

(b) Like any fundamental matrix, $\Phi = e^{Ax}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x} e^{Ax} = Ae^{Ax}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}x} e^{Ax}\right]_{x=0} = \left[\begin{bmatrix} 3e^x - 4e^{2x} & -6e^x + 12e^{2x} \\ e^x - 2e^{2x} & -2e^x + 6e^{2x} \end{bmatrix}\right]_{x=0} = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$. □

Problem 8. (4 points) Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(2) = 0$ in the case $\lambda > 0$.

Solution. Write $\lambda = \rho^2$. Then $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using this, $y(2) = B\sin(2\rho) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(2\rho) = 0$. This happens if $2\rho = n\pi$ for an integer n.

Consequently, we find the eigenvalues $\lambda = \left(\frac{n\pi}{2}\right)^2$, where $n = 1, 2, 3, \ldots$ (we exclude $n = 0$ because $\rho > 0$), with corresponding eigenfunctions $y(x) = \sin\frac{n\pi x}{2}$. $\square$

Problem 9. (3 points) Let $y(x)$ be the unique solution to the IVP $y'' = x + 3y^2$, $y(0) = 2$, $y'(0) = 1$.

Determine the first several terms (up to x^3) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 + 3y(0)^2 = 12$.

Differentiating both sides of the DE, we obtain $y''' = 1 + 6yy'$. In particular, $y'''(0) = 1 + 6 \cdot 2 \cdot 1 = 13$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots = 2 + x + 6x^2 + \frac{13}{6}x^3 + \ldots$ $\square$

Solution. (plug in power series) Taking into account the initial conditions, $y = 2 + x + a_2x^2 + a_3x^3 + \ldots$.

Therefore, $y'' = 2a_2 + 6a_3x + \ldots$

On the other hand, $y^2 = 4 + 4x + \ldots$

Equating coefficients of y'' and $x + 3y^2$, we find $2a_2 = 12$, $6a_3 = 1 + 3 \cdot 4 = 13$.

So $a_2 = 6$, $a_3 = \frac{13}{6}$ and, hence, $y(x) = 2 + x + 6x^2 + \frac{13}{6}x^3 + \ldots$ $\square$

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(extra scratch paper)