Midterm #2 – Practice

MATH 332 — Differential Equations II
Tuesday, Nov 12

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Suppose that the matrix A satisfies $e^{Ax} = \frac{1}{7}\begin{bmatrix} e^{-9x}+6e^{-2x} & -2e^{-9x}+2e^{-2x} \\ -3e^{-9x}+3e^{-2x} & 6e^{-9x}+e^{-2x} \end{bmatrix}$.

(a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 3e^x \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) What is A?

Solution.

(a) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14e^{-2x} \\ 7e^{-2x} \end{bmatrix} = \begin{bmatrix} 2e^{-2x} \\ e^{-2x} \end{bmatrix}$

(b) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$. We compute:

$\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \int_0^x \frac{1}{7}\begin{bmatrix} e^{9t}+6e^{2t} & -2e^{9t}+2e^{2t} \\ -3e^{9t}+3e^{2t} & 6e^{9t}+e^{2t} \end{bmatrix}\begin{bmatrix} 0 \\ 3e^t \end{bmatrix}\mathrm{d}t = \frac{3}{7}\int_0^x \begin{bmatrix} -2e^{10t}+2e^{3t} \\ 6e^{10t}+e^{3t} \end{bmatrix}\mathrm{d}t = \frac{3}{7}\begin{bmatrix} -\frac{1}{5}e^{10x}+\frac{2}{3}e^{3x}-\frac{7}{15} \\ \frac{3}{5}e^{10x}+\frac{1}{3}e^{3x}-\frac{14}{15} \end{bmatrix}$
$= \frac{1}{35}\begin{bmatrix} -3e^{10x}+10e^{3x}-7 \\ 9e^{10x}+5e^{3x}-14 \end{bmatrix}$

Hence, $e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \frac{1}{7}\begin{bmatrix} e^{-9x}+6e^{-2x} & -2e^{-9x}+2e^{-2x} \\ -3e^{-9x}+3e^{-2x} & 6e^{-9x}+e^{-2x} \end{bmatrix}\frac{1}{35}\begin{bmatrix} -3e^{10x}+10e^{3x}-7 \\ 9e^{10x}+5e^{3x}-14 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 3e^{-9x}-10e^{-2x}+7e^{x} \\ -9e^{-9x}-5e^{-2x}+14e^{x} \end{bmatrix}$.

Finally, $\boldsymbol{y}(x) = \begin{bmatrix} 2e^{-2x} \\ e^{-2x} \end{bmatrix} + \frac{1}{35}\begin{bmatrix} 3e^{-9x}-10e^{-2x}+7e^{x} \\ -9e^{-9x}-5e^{-2x}+14e^{x} \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 3e^{-9x}+60e^{-2x}+7e^{x} \\ -9e^{-9x}+30e^{-2x}+14e^{x} \end{bmatrix}$.

(c) Like any fundamental matrix, $\Phi = e^{Ax}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax} = Ae^{Ax}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}\right]_{x=0} = \left[\frac{1}{7}\begin{bmatrix} -9e^{-9x}-12e^{-2x} & 18e^{-9x}-4e^{-2x} \\ 27e^{-9x}-6e^{-2x} & -54e^{-9x}-2e^{-2x} \end{bmatrix}\right]_{x=0} = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}$. □

Problem 2.

(a) Suppose $y(x)$ has the power series $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. How can we compute the a_n from $y(x)$?

(b) Suppose $f(t)$ has the Fourier series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right)$.

How can we compute the a_n and b_n from $f(t)$?

Solution.

(a) $a_n = \dfrac{y^{(n)}(x_0)}{n!}$

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(b) The Fourier coefficients a_n, b_n can be computed as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t. \qquad \square$$

Problem 3. Spell out the power series (around $x=0$) of the following functions.

(a) e^x

(b) $\sin(3x^2)$

(c) $\dfrac{5}{1+7x^2}$

Solution.

(a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, we have $\sin(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{4n+2}$.

(c) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{5}{1+7x^2} = 5\sum_{n=0}^{\infty} (-7x^2)^n = 5\sum_{n=0}^{\infty} (-7)^n x^{2n}$. $\square$

Problem 4. Let $y(x)$ be the unique solution to the IVP $y'' = x + 2y^3$, $y(0) = 1$, $y'(0) = 2$.

Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 + 2y(0)^3 = 2$.

Differentiating both sides of the DE, we obtain $y''' = 1 + 6y^2\, y'$. In particular, $y'''(0) = 13$.

Continuing, $y^{(4)} = 12y(y')^2 + 6y^2 y''$ so that $y^{(4)}(0) = 12 \cdot 1 \cdot 2^2 + 6 \cdot 1^2 \cdot 2 = 60$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \ldots = 1 + 2x + x^2 + \frac{13}{6}x^3 + \frac{5}{2}x^4 + \ldots$ $\square$

Solution. (plug in power series) Taking into account the initial conditions, $y = 1 + 2x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$.

Therefore, $y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots$

On the other hand, $y^3 = 1 + 6x + (3a_2 + 12)x^2 + \ldots$

Equating coefficients of y'' and $x + 2y^3$, we find $2a_2 = 2$, $6a_3 = 1 + 2 \cdot 6 = 13$, $12a_4 = 2(3a_2 + 12)$.

So $a_2 = 1$, $a_3 = \frac{13}{6}$, $a_4 = \frac{1}{2}a_2 + 2 = \frac{5}{2}$ and, hence, $y(x) = 1 + 2x + x^2 + \frac{13}{6}x^3 + \frac{5}{2}x^4 + \ldots$ $\square$

Problem 5. Consider the DE $y'' = x(x^2+7)\, y' + (x^2+3)y$.

Derive a recursive description of a power series solutions $y(x)$.

Solution. Let us spell out the power series for y, x^2y, xy', x^3y', y'':

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$x y'(x) = \sum_{n=1}^{\infty} n a_n x^n$$

$$x^3 y'(x) = \sum_{n=1}^{\infty} n a_n x^{n+2} = \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Hence, the DE becomes:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=3}^{\infty} (n-2) a_{n-2} x^n + 7\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n + 3\sum_{n=0}^{\infty} a_n x^n.$$

We compare coefficients of x^n:

- $n=0$: $2a_2 = 3a_0$, so that $a_2 = \frac{3}{2} a_0$.
- $n=1$: $6a_3 = 7a_1 + 3a_1$, so that $a_3 = \frac{5}{3} a_1$.
- $n=2$: $12a_4 = 14a_2 + a_0 + 3a_2$, so that $a_4 = \frac{1}{12} a_0 + \frac{17}{12} a_2 = \frac{1}{12} a_0 + \frac{17}{12} \frac{3}{2} a_0 = \frac{53}{24} a_0$.
- $n \geqslant 3$: $(n+2)(n+1) a_{n+2} = (n-2) a_{n-2} + 7n a_n + a_{n-2} + 3a_n$

  $a_{n+2} = \frac{7n+3}{(n+2)(n+1)} a_n + \frac{n-1}{(n+2)(n+1)} a_{n-2}$.

  Equivalently, for $n \geqslant 5$, $a_n = \frac{7n-11}{n(n-1)} a_{n-2} + \frac{n-3}{n(n-1)} a_{n-4}$.

In conclusion, the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_2 = \frac{3}{2} a_0, \quad a_3 = \frac{5}{3} a_1, \quad a_4 = \frac{53}{24} a_0, \quad a_n = \frac{7n-11}{n(n-1)} a_{n-2} + \frac{n-3}{n(n-1)} a_{n-4} \quad \text{for } n \geqslant 5.$$

(The values a_0 and a_1 are the initial conditions.)

Comment. The formula for a_n also holds for $n=4$. Can you see why? □

Problem 6. Find a minimum value for the radius of convergence of a power series solution to $(4x^2+1) y'' = \frac{y}{x+1}$ at $x=3$.

Solution. Rewriting the DE as $y'' - \frac{1}{(x+1)(4x^2+1)} y = 0$, we see that the singular points are $x = \pm i/2, -1$.

Note that $x=3$ is an ordinary point of the DE and that the distance to the nearest singular point is $|3 - (\pm i/2)| = \sqrt{3^2 + (1/2)^2} = \frac{1}{2}\sqrt{37} \approx 3.04$ (the distance to -1 is $|3-(-1)| = 4$).

Hence, the DE has power series solutions about $x=3$ with radius of convergence at least $\frac{1}{2}\sqrt{37}$. □

Problem 7. Derive a recursive description of the power series for $y(x) = \dfrac{1}{1-2x-5x^2}$.

Solution. Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} 1 = (1-2x-5x^2)\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} - 5\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n - 5\sum_{n=2}^{\infty} a_{n-2} x^n. \end{aligned}$$

We compare coefficients of x^n:

- $n=0$: $\quad 1 = a_0$.
- $n=1$: $\quad 0 = a_1 - 2a_0$, so that $a_1 = 2a_0 = 2$.
- $n \geqslant 2$: $\quad 0 = a_n - 2a_{n-1} - 5a_{n-2}$ or, equivalently, $a_n = 2a_{n-1} + 5a_{n-2}$.

In conclusion, the power series $\frac{1}{1-2x-5x^2} = \sum_{n=0}^{\infty} a_n x^n$ is recursively determined by

$$a_0 = 1, \quad a_1 = 2, \quad a_n = 2a_{n-1} + 5a_{n-2} \quad \text{for } n \geqslant 2.$$

□

Problem 8. A mass-spring system is described by the equation

$$my'' + y = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

(a) For which m does resonance occur?

(b) Find the general solution when $m = 1/9$.

Solution.

(a) The roots of $p(D) = mD^2 + 1$ are $\pm i/\sqrt{m}$, so that that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m} = n/3$ for some $n \in \{1, 3, 5, \ldots\}$. Equivalently, resonance occurs if $m = 9/n^2$ for an odd integer $n \geqslant 1$ (that is, $m = 9, 1, 9/25, 9/49, \ldots$).

(b) In this case, the natural frequency is 3 and we have resonance because $3 = n/3$ for $n = 9$. For $n \neq 9$ we solve

$$\frac{1}{9}y'' + y = \frac{1}{n^2}\sin\left(\frac{nt}{3}\right).$$

This has a solution of the form $y_p = A\cos\left(\frac{nt}{3}\right) + B\sin\left(\frac{nt}{3}\right)$ where A, B are undetermined. Plugging into the DE:

$$\frac{1}{9}y_p'' + y_p = A\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\cos\left(\frac{nt}{3}\right) + B\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\sin\left(\frac{nt}{3}\right) \overset{!}{=} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$$

It follows that $A = 0$ (we could have seen that coming...) and

$$B = \frac{1}{n^2\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)} = \frac{81}{n^2(81-n^2)}, \qquad y_p = \frac{81}{n^2(81-n^2)}\sin\left(\frac{nt}{3}\right).$$

The case $n = 9$ has to be done separately: because of resonance there now exists a solution of the form

$$y_p = At\cos(3t) + Bt\sin(3t).$$

Plugging into the DE:

$$\frac{1}{9}y_p'' + y_p = \frac{2}{3}B\cos(3t) - \frac{2}{3}A\sin(3t) \overset{!}{=} \frac{1}{81}\sin(3t)$$

It follows that $B = 0$ and $A = -\frac{1}{54}$. So $y_p = -\frac{1}{54}t\cos(3t)$. By superposition it follows that

$$\frac{1}{9}y'' + y = \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{1}{n^2}\sin\left(\frac{nt}{3}\right) \qquad \text{has solution} \qquad y_p = -\frac{1}{54}t\cos(3t) + \sum_{\substack{n=1\\ n\text{ odd},\, n\neq 9}} \frac{81}{n^2(81-n^2)}\sin\left(\frac{nt}{3}\right).$$

The general solution is $y(t) = y_p(t) + A\cos(3t) + B\sin(3t)$. □

Problem 9. Consider the function $f(t) = 2(1-t)$, defined for $t \in [0,1]$.

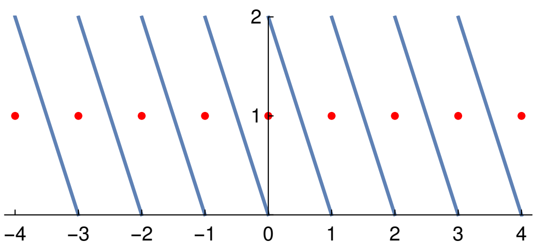
(a) Sketch the Fourier series of $f(t)$ for $t \in [-4,4]$.

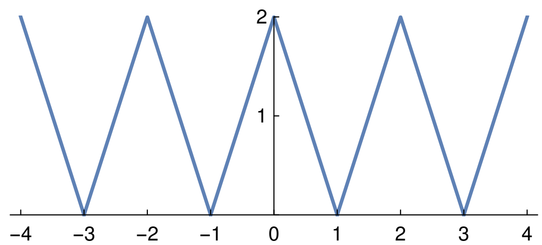
(b) Sketch the Fourier cosine series of $f(t)$ for $t \in [-4,4]$.

(c) Sketch the Fourier sine series of $f(t)$ for $t \in [-4,4]$.

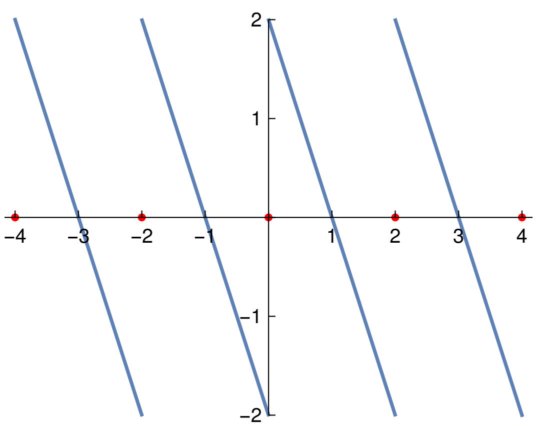
In each sketch, carefully mark the values of the Fourier series at discontinuities.

Solution. The Fourier series (i.e. the 2-periodic extension) as well as the Fourier cosine series (i.e. the 4-periodic even extension):





The Fourier sine series (i.e. the 4-periodic odd extension):



In each sketch, the function values at discontinuities are marked in red. □

Problem 10. Compute the Fourier sine series of the function $f(t)$, defined for $t \in (0, L)$, with $f(t) = 3$.

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Solution. The odd $2L$-periodic extension of $f(t)$ takes the values $f(t) = \begin{cases} -3, & \text{for } t \in (-L, 0). \\ +3, & \text{for } t \in (0, L). \end{cases}$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t = \frac{2}{L}\int_0^L 3\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t = \frac{6}{L}\left[-\frac{L}{\pi n}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L = \frac{6}{\pi n}[1-\cos(n\pi)] \\ &= \frac{6}{\pi n}[1-(-1)^n] = \begin{cases} \frac{12}{\pi n}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus the Fourier sine series is:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{12}{\pi n}\sin\left(\frac{n\pi t}{L}\right)$$

□

Problem 11. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

(Make sure to consider all cases.)

Solution. To solve this eigenvalue problem, we distinguish three cases:

- $\boldsymbol{\lambda < 0}$**.** Then, the roots are the real numbers $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

- $\boldsymbol{\lambda = 0}$**.** Now, the general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

- $\boldsymbol{\lambda > 0}$**.** Now, the roots are $\pm i\sqrt{\lambda}$ and $y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$. $y'(0) = B\sqrt{\lambda} = 0$ implies $B = 0$. Then $y(3) = A\cos(3\sqrt{\lambda}) = 0$. Note that $\cos(3\sqrt{\lambda}) = 0$ is true if and only if $3\sqrt{\lambda} = \frac{(2n+1)\pi}{2}$ for some integer n. In that case, $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, this means that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ (why did we include $n = 0$ but excluded $n = -1, -2, \ldots$?!) with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$. □

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