Midterm #1

MATH 332 — Differential Equations II
Tuesday, Oct 1

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (3 points) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' - 5y_2 \\ y_2'' &= y_1' - y_2' + 3y_1 \end{aligned} \qquad y_1(0) = -2,\ y_1'(0) = 1,\ y_2(0) = 0,\ y_2'(0) = 3$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 3 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

□

Problem 2. (3 points) Determine a (homogeneous linear) recurrence equation satisfied by $a_n = (n+2)3^n - 7$.

Write the recurrence in explicit form (for instance, $a_{n+2} = a_{n+1} + a_n$ for the Fibonacci numbers).

Solution. $a_n = (n+2)3^n - 7$ is a solution of $p(N)a_n = 0$ if and only if 3 (repeated two times) and 1 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-3)^2(N-1) = N^3 - 7N^2 + 15N - 9$.

The corresponding recurrence is $a_{n+3} = 7a_{n+2} - 15a_{n+1} + 9a_n$. □

Problem 3. (9 points) Let $M = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}$.

(a) Compute e^{Mx}.

(b) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & 4 \\ 8 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(1-\lambda) - 32 = \lambda^2 - 6\lambda - 27 = (\lambda+3)(\lambda-9)$$

Armin Straub
straub@southalabama.edu

Hence, the eigenvalues are $\lambda = -3$ and $\lambda = 9$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = -3$, we need to solve $\begin{bmatrix} 8 & 4 \\ 8 & 4 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = -3$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 9$, we need to solve $\begin{bmatrix} -4 & 4 \\ 8 & -8 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 9$.

Hence, a fundamental matrix solution is $\Phi = \begin{bmatrix} -e^{-3x} & e^{9x} \\ 2e^{-3x} & e^{9x} \end{bmatrix}$.

Note that $\Phi(0) = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -e^{-3x} & e^{9x} \\ 2e^{-3x} & e^{9x} \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-3x} + 2e^{9x} & -e^{-3x} + e^{9x} \\ -2e^{-3x} + 2e^{9x} & 2e^{-3x} + e^{9x} \end{bmatrix}.$$

(b) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-3x} + 2e^{9x} & -e^{-3x} + e^{9x} \\ -2e^{-3x} + 2e^{9x} & 2e^{-3x} + e^{9x} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -e^{-3x} + e^{9x} \\ 2e^{-3x} + e^{9x} \end{bmatrix}$. □

Problem 4. (1+4+1 points) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -2$, $a_1 = 6$.

(a) The next two terms are $a_2 = \boxed{}$ and $a_3 = \boxed{}$.

(b) A Binet-like formula for a_n is $a_n = \boxed{}$, and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \boxed{}$.

Solution.

(a) $a_2 = 6$, $a_3 = 30$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 2N - 3$ has roots $3, -1$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = -2$, $a_1 = 3\alpha_1 - \alpha_2 = 6$.

Solving, we find $\alpha_1 = 1$ and $\alpha_2 = -3$ so that, in conclusion, $a_n = 3^n - 3(-1)^n$.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$. □

Problem 5. (2 points) Consider a homogeneous linear differential equation with constant real coefficients which has order 5. Suppose $y(x) = 4x^2 e^{-x} + e^{3x}\sin(2x)$ is a solution. Write down the general solution.

Solution. $y(x) = 4x^2 e^{-x} + e^{3x}\sin(2x)$ is a solution of $p(D)y = 0$ if and only if $-1, -1, -1, 3 \pm 2i$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 5, there can be no further roots.

The general solution of this DE is $y(x)4 = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{3x}\cos(2x) + c_5 e^{3x}\sin(2x)$. □

Problem 6. (4 points) Find the general solution to $y'' - 4y = 3e^{2x} + 5$.

Solution. The characteristic polynomial $p(D) = D^2 - 4$ of the associated homogeneous DE has "old" roots ± 2.

The "new" roots coming from $3e^{2x} + 5$ are $0, 2$. Hence, there has to be a particular solution of the form $y_p = Axe^{2x} + B$. To find the values of A, B, we plug into the DE.

$y_p' = A(2x + 1)e^{2x}$

$y_p'' = A(4x + 4)e^{2x}$

$y_p'' - 4y_p = 4Ae^{2x} - 4B \overset{!}{=} 3e^{2x} + 5$

Consequently, $A = \frac{3}{4}$, $B = -\frac{5}{4}$.

Hence, the general solution is $y(x) = \left(c_1 + \frac{3}{4}x\right)e^{2x} - \frac{5}{4} + c_2 e^{-2x}$. □

Problem 7. (3 points) Let y_p be any solution to the inhomogeneous linear differential equation $x^2 y'' - y = e^{2x}$. Find a homogeneous linear differential equation which y_p solves. *Hint:* Do not attempt to solve the DE.

Solution. To kill e^{2x}, we apply $D - 2$ to both sides of the DE $x^2 y'' - y = e^{2x}$.

The result is the homogeneous linear DE $(x^2 y''' + 2xy'' - y') - 2(x^2 y'' - y) = x^2 y''' + (2x - 2x^2)y'' - y' + 2y = 0$. □

Armin Straub
straub@southalabama.edu

(extra scratch paper)