Sketch of Lecture 18

Tue, 10/29/2019

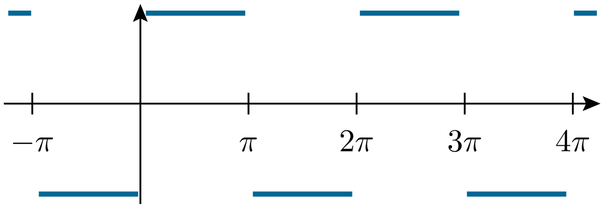
Definition 106. Let $L > 0$. $f(t)$ is **L-periodic** if $f(t + L) = f(t)$ for all t. The smallest such L is called the **(fundamental) period** of f.

Example 107. The fundamental period of $\cos(nt)$ is $2\pi/n$.

Example 108. The trigonometric functions $\cos(nt)$ and $\sin(nt)$ are 2π-periodic for any integer n. And so are their linear combinations. (In other words, 2π-periodic functions form a vector space.)

Example 109. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

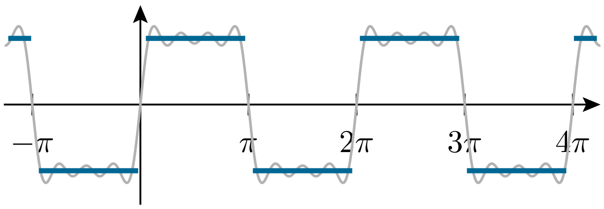
$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$



Solution. We compute $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0$, as well as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t + \int_{0}^{\pi}\cos(nt)\mathrm{d}t\right] = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\mathrm{d}t = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t + \int_{0}^{\pi}\sin(nt)\mathrm{d}t\right] = \frac{2}{\pi n}[1 - \cos(n\pi)] \\ &= \frac{2}{\pi n}[1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion, $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(nt) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right)$.



Observation. The coefficients a_n are zero for all n if and only if $f(t)$ is odd.

Comment. The value of $f(t)$ for $t = -\pi, 0, \pi$ is irrelevant to the computation of the Fourier series. They are chosen so that $f(t)$ is equal to the Fourier series for all t (recall that, at a jump discontinuity t, the Fourier series converges to the average $\frac{f(t^-) + f(t^+)}{2}$).

Comment. Plot the (sum of the) first few terms of the Fourier series. What do you observe? The "overshooting" is known as the **Gibbs phenomenon**: `https://en.wikipedia.org/wiki/Gibbs_phenomenon`

Comment. Set $t=\frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series $\pi=4[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots]$. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point ($3.14159265\ldots721134999999\ldots$), we would (roughly) need $1/n<10^{-768}$, or $n>10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

Remark. Convergence of such series is not obvious! Recall, for instance, that the (odd part of) the harmonic series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ diverges.

There is nothing special about 2π-periodic functions considered last time (except that $\cos(t)$ and $\sin(t)$ have fundamental period 2π). Note that $\cos(\pi t/L)$ and $\sin(\pi t/L)$ have period $2L$.

If $f(t)$ has period $2L$, then $\tilde{f}(x):=f\left(\frac{L}{\pi}t\right)$ has period 2π. Therefore Theorem 105 implies the following:

Theorem 110. Every* $2L$-periodic function f can be written as a **Fourier series**

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right).$$

Technical detail*: f needs to be, e.g., piecewise smooth.

Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$.

The **Fourier coefficients** a_n, b_n are unique and can be computed as

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t,\qquad b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t.$$

Example 111. Find the Fourier series of the 2-periodic function $g(t)=\begin{cases}-1 & \text{for } t\in(-1,0)\\ +1 & \text{for } t\in(0,1)\\ 0 & \text{for } t=-1,0,1\end{cases}$.

Solution. Instead of computing from scratch, we can use the fact that $g(t)=f(\pi t)$, with f as in the previous example, to get $g(t)=f(\pi t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t)$.

Theorem 112. If $f(t)$ is **continuous** and $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t)=\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}b_n\cos\left(\frac{n\pi t}{L}\right)-\frac{n\pi}{L}a_n\sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Example 113. Let $h(t)$ be the 2-periodic function with $h(t)=\begin{cases}-t & \text{for } t\in(-1,0)\\ +t & \text{for } t\in(0,1)\end{cases}$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n=0$. Computing a_n is left as an exercise.

Solution. Note that $h(t)$ is continuous and $h'(t)=g(t)$, with $g(t)$ as in Example 111. Hence, we can apply Theorem 112 to conclude

$$h'(t)=g(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t)\quad\Longrightarrow\quad h(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\left(-\frac{1}{\pi n}\right)\cos(n\pi t)+C,$$

where $C=\frac{a_0}{2}=\frac{1}{2}\int_{-1}^{1}h(t)\mathrm{d}t=\frac{1}{2}$ is the constant of integration. Thus, $h(t)=\frac{1}{2}-\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi^2 n^2}\cos(n\pi t)$.

Remark. Note that $t=0$ in the last Fourier series, gives us $\frac{\pi^2}{8}=\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$. As an exercise, you can try to find from here the fact that $\sum_{n\geqslant 1}\frac{1}{n^2}=\frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n\geqslant 1}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3),\zeta(5),\ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.

Example 114. (caution!) The function $g(t)$, from Example 111, is not continuous. For all values, except the discontinuities, we have $g'(t)=0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n\text{ odd}}\cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t=0$). This illustrates that we cannot apply Theorem 112 because of the missing continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

Fourier series and linear differential equations

In the following examples, we look at inhomogeneous linear DEs $p(D)y=f(t)$ where $f(t)$ is a periodic function that can be expressed as a Fourier series.

Example 115. Consider the linear DE $my''+ky=\cos(\omega t)$. For which (external) **frequencies** ω does **resonance** occur?

Solution. The roots of $p(D)=mD^2+k$ are $\pm i\sqrt{k/m}$. Correspondingly, the solutions of the homogeneous equation $my''+ky=0$ are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0=\sqrt{k/m}$ (ω_0 is called the **natural frequency** of the DE). Resonance occurs in the case $\omega=\omega_0$ (overlapping roots).

Review. If $\omega\neq\omega_0$, then there is particular solution of the form $y_p(t)=A\cos(\omega t)+B\sin(\omega t)$ (for specific values of A and B). The general solution is $y(t)=A\cos(\omega t)+B\sin(\omega t)+C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)$, which is a bounded function of t. In contrast, if $\omega=\omega_0$, then general solution is $y(t)=(C_1+At)\cos(\omega_0 t)+(C_2+Bt)\sin(\omega_0 t)$ and this function is unbounded.

Comment. The inhomogeneous equation $my''+ky=F(t)$ describes the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$.

Example 116. A mass-spring system is described by the DE $2y''+32y=\sum_{n=1}^{\infty}\frac{\cos(n\omega t)}{n^2+1}$.

For which ω does resonance occur?

Solution. The roots of $p(D)=2D^2+32$ are $\pm 4i$, so that that the natural frequency is 4. Resonance therefore occurs if 4 equals $n\omega$ for some $n\in\{1,2,3,\ldots\}$. Equivalently, resonance occurs if $\omega=4/n$ for some $n\in\{1,2,3,\ldots\}$.

Example 117. A mass-spring system is described by the DE $my''+y=\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{nt}{3}\right)$.

For which m does resonance occur?

Solution. The roots of $p(D)=mD^2+1$ are $\pm i/\sqrt{m}$, so that that the natural frequency is $1/\sqrt{m}$. Resonance therefore occurs if $1/\sqrt{m}=n/3$ for some $n\in\{1,2,3,\ldots\}$. Equivalently, resonance occurs if $m=9/n^2$ for some $n\in\{1,2,3,\ldots\}$.