Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can written as

$$\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x).$$

Note. In general, A depends on x. In other words, the DE is allowed to have nonconstant coefficients.

Review. We showed in Theorem 17 that $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} \mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Amazingly (or, maybe, by now, not surprisingly), the same arguments with the same result apply to systems of linear equations:

Theorem 71. (variation of constants) $\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x)$ has the particular solution

$$\boldsymbol{y}_p(x) = \Phi(x) \int \Phi(x)^{-1} \boldsymbol{f}(x) \mathrm{d}x,$$

where $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$.

Proof. We can find this formula in the same manner as we did in Theorem 17:

Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(x)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$. Plugging into the DE, we get:

$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \overset{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.

Hence, $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.

The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x$, which gives the claimed formula for $\boldsymbol{y}_p$. $\square$

Example 72. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3x} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}$

Using $\det(\Phi(x)) = 5e^{3x}$, we find $\Phi(x)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{x} & -3e^{x} \\ e^{-4x} & e^{-4x} \end{bmatrix}$.

Hence, $\Phi(x)^{-1}\boldsymbol{f}(x) = \frac{2}{5}\begin{bmatrix} 3e^{4x} \\ -e^{-x} \end{bmatrix}$ and $\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(x) = \Phi(x)\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3x}$.

In the special case that $\Phi(x) = e^{Ax}$, some things become easier. For instance, $\Phi(x)^{-1} = e^{-Ax}$. Also, we can just write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(x)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$.

Armin Straub
straub@southalabama.edu

Example 73. Suppose that the matrix A satisfies $e^{Ax} = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}$.

(a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^x \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) What is A?

Solution.

(a) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix}$

(b) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$. We compute:

$\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \int_0^x \begin{bmatrix} 2e^{-2t} - e^{-3t} & -2e^{-2t} + 2e^{-3t} \\ e^{-2t} - e^{-3t} & -e^{-2t} + 2e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^t \end{bmatrix}\mathrm{d}t = \int_0^x \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix}\mathrm{d}t = \begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix}$

Hence, $e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}\begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix} = \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix}$.

Finally, $\boldsymbol{y}(x) = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix} + \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix} = \begin{bmatrix} 2e^x - 6e^{2x} + 5e^{3x} \\ -3e^{2x} + 5e^{3x} \end{bmatrix}$.

(c) Like any fundamental matrix, $\Phi = e^{Ax}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax} = Ae^{Ax}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}\right]_{x=0} = \left[\begin{bmatrix} 4e^{2x} - 3e^{3x} & -4e^{2x} + 6e^{3x} \\ 2e^{2x} - 3e^{3x} & -2e^{2x} + 6e^{3x} \end{bmatrix}\right]_{x=0} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

Modelling

Example 74. Consider two brine tanks. Tank T_1 is filled with 24gal water containing 3lb salt, and tank T_2 with 9gal pure water.

- T_1 is filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.

$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

This is an IVP that we can solve (with some work)! Do it! Skipping most work, we find:

- $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix} = \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$

Note. We could have found a particular solution without calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. Of course, we can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 3lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.