Systems of recurrence equations

Example 55. Write the (second-order) RE $a_{n+2}=a_{n+1}+2a_n$, with $a_0=1$, $a_1=8$, as a system of (first-order) recurrences.

Solution. Write $b_n=a_{n+1}$.

Then, $a_{n+2}=a_{n+1}+2a_n$ translates into the first-order system $\begin{cases} a_{n+1}=b_n \\ b_{n+1}=2a_n+b_n \end{cases}$.

Let $\boldsymbol{a}_n=\begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1}=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{a}_n$, with $\boldsymbol{a}_0=\begin{bmatrix} 1 \\ 8 \end{bmatrix}$.

Comment. Consequently, $\boldsymbol{a}_n=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n\boldsymbol{a}_0=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n\begin{bmatrix} 1 \\ 8 \end{bmatrix}$. Solving (systems of) REs is equivalent to computing powers of matrices!

Example 56. Determine the general solution to $\boldsymbol{a}_{n+1}=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{a}_n$.

Solution. In the previous example, we obtained this system from the RE $a_{n+2}=a_{n+1}+2a_n$, which we know (do it!) has solutions $a_n=2^n$ and $a_n=(-1)^n$ (which combine to the general solution $a_n=C_1\cdot 2^n+C_2\cdot(-1)^n$).

Correspondingly, $\boldsymbol{a}_{n+1}=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{a}_n$ has solutions $\boldsymbol{a}_n=\begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}$ and $\boldsymbol{a}_n=\begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$.

These combine to the general solution $\boldsymbol{a}_n=C_1\begin{bmatrix} 2^n \\ 2^{n+1} \end{bmatrix}+C_2\begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}=\begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

We call $\Phi_n=\begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}$ a **fundamental matrix (solution)**. The general solution is $\Phi_n\boldsymbol{c}$ with $\boldsymbol{c}=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Observations.

(a) The columns of Φ_n are (independent) solutions of the system.

(b) Φ_n solves the RE itself: $\Phi_{n+1}=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\Phi_n$.

[Spell this out in this example! That Φ_n solves the RE follows from the definition of matrix multiplication.]

(c) It follows that $\Phi_n=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n\Phi_0$. Equivalently, $\Phi_n\Phi_0^{-1}=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n$. (See next example!)

Matrix powers M^n can be computed by diagonalizing the matrix M (if you have taken linear algebra classes, you might have seen this).

We just saw that being able to compute matrix powers is equivalent to solving systems of recurrences. In the next example, we use this connection to compute some matrix powers.

(a way to compute powers of a matrix M)
Compute a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.
Then $M^n=\Phi_n\Phi_0^{-1}$.

Example 57. Compute M^n for $M=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution. We already observed that $\Phi_n=\begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}$ is a fundamental matrix solution Φ_n of $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

We have $\Phi_0^{-1}=\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1}=\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Hence,

$$M^n=\Phi_n\Phi_0^{-1}=\begin{bmatrix} 2^n & (-1)^n \\ 2^{n+1} & (-1)^{n+1} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 2^n+2(-1)^n & 2^n-(-1)^n \\ 2^{n+1}+2(-1)^{n+1} & 2^{n+1}-(-1)^{n+1} \end{bmatrix}.$$

Note. M^n is a fundamental matrix solution of $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ itself.

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Example 58. Let $M=\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) Let us look for solutions of the form $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$ (where $\boldsymbol{v}=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$). Note that $\boldsymbol{a}_{n+1}=\boldsymbol{v}\lambda^{n+1}=\lambda\boldsymbol{a}_n$.
Plugging into $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ we find $\boldsymbol{v}\lambda^{n+1}=M\boldsymbol{v}\lambda^n$.
Cancelling λ^n (just a number!), this simplifies to $\lambda\boldsymbol{v}=M\boldsymbol{v}$.
In other words, $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.
We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=-2$.
Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}3^n+C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-2)^n$.

(b) The corresponding fundamental matrix solution is $\Phi_n=\begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}$.
[Note that our general solution is precisely $\Phi_n\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Note that $\Phi_0=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi_0^{-1}=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$M^n=\Phi_n\Phi_0^{-1}=\begin{bmatrix} 2\cdot 3^n & (-2)^n \\ 3^n & (-2)^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 2\cdot 3^n-(-2)^n & -2\cdot 3^n+2(-2)^n \\ 3^n-(-2)^n & -3^n+2(-2)^n \end{bmatrix}.$$

To solve $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$
- If there is enough eigenvectors, these combine to the general solution.

Comment. If there is not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$, we also need to look for solutions of the type $\boldsymbol{a}_n=(\boldsymbol{v}n+\boldsymbol{w})\lambda^n$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

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