Midterm #2

MATH 238 — Differential Equations I
Friday, Apr 11, 2025

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 36 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points) The mixtures in two tanks T_1, T_2 are kept uniform by stirring. Brine containing 3 lb of salt per gallon enters T_1 at a rate of 3 gal/min, while brine containing 2 lb of salt per gallon enters T_2 at a rate of 4gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 1 gal/min, and also from T_2 into T_1 at a rate of 2 gal/min. Initially, tank T_1 is filled with 10 gal water containing 5 lb salt, and tank T_2 with 20 gal pure water.

Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions. (Do *not* attempt to solve the system.)

Solution. Note that after t minutes, T_1 contains $10+3t-t+2t=10+4t$ gal of solution while T_2 contains $20+4t+t-2t=20+3t$ gal of solution. In the time interval $[t, t+\Delta t]$, we have:

$$\begin{array}{lcl} \Delta y_1 \approx 3\cdot 3\cdot \Delta t - 1\cdot \frac{y_1}{10+4t}\cdot \Delta t + 2\cdot \frac{y_2}{20+3t}\cdot \Delta t & \Longrightarrow & y_1' = 9 - \frac{1}{10+4t} y_1 + \frac{2}{20+3t} y_2 \\ \Delta y_2 \approx 4\cdot 2\cdot \Delta t + 1\cdot \frac{y_1}{10+4t}\cdot \Delta t - 2\cdot \frac{y_2}{20+3t}\cdot \Delta t & \Longrightarrow & y_2' = 8 + \frac{1}{10+4t} y_1 - \frac{2}{20+3t} y_2 \end{array}$$

The initial conditions are $y_1(0)=5,\ y_2(0)=0$.

Optional: in matrix form, writing $\boldsymbol{y}=(y_1, y_2)$, this takes the form

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{1}{10+4t} & \frac{2}{20+3t} \\ \frac{1}{10+4t} & -\frac{2}{20+3t} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 9 \\ 8 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Problem 2. (4 points) Consider the following system of initial value problems:

$$\begin{array}{l} y_1'' + 3y_2 = y_1' + 5 \\ y_2'' + 2y_1 = 7y_2' \end{array} \qquad y_1(0)=4,\ y_1'(0)=-1,\ y_2(0)=0,\ y_2'(0)=7$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y} + \boldsymbol{f},\ \boldsymbol{y}(0) = \boldsymbol{c}$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 \\ -2 & 0 & 0 & 7 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 7 \end{bmatrix}.$$

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Problem 3. (3 points) The position $y(t)$ of a certain mass on a spring is described by $my'' + 5y = \cos(t) - 2\sin(3t)$. For which values of m, if any, does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{5}{m}}$ while the external frequencies are 1 and 3. Resonance therefore occurs if $\sqrt{\frac{5}{m}} = 1$ or $\sqrt{\frac{5}{m}} = 3$. Equivalently, if $m = 5$ or $m = \frac{5}{9}$.

Problem 4. (10 points) Determine the general solution of the following system: $\begin{array}{rcl} y_1' &=& y_1 + y_2 \\ y_2' &=& 3y_1 - y_2 + 9e^{-x} \end{array}$

Solution. Using $y_2 = y_1' - y_1$ (from the first equation) in the second equation, we get $y_1'' - y_1' = 3y_1 - (y_1' - y_1) + 9e^{-x}$.

Simplified, this is $y_1'' - 4y_1 = 9e^{-x}$. This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots for the homogeneous DE are ± 2, while the root for the inhomogeneous part is 1, there must a particular solution of the form $y_1 = Ae^{-x}$. For this y_1, $y_1'' - 4y_1 = (1 - 4)Ae^{-x} = -3Ae^{-x} \overset{!}{=} 9e^{-x}$. Hence, $A = -3$ and the particular solution is $y_1 = -3e^{-x}$. The corresponding general solution is $y_1 = -3e^{-x} + C_1e^{2x} + C_2e^{-2x}$.

Correspondingly, $y_2 = 3e^{x} + 2C_1e^{2x} - 2C_2e^{-2x} - (-3e^{-x} + C_1e^{2x} + C_2e^{-2x}) = 6e^{-x} + C_1e^{2x} - 3C_2e^{-2x}$.

Problem 5. (4 points) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 9\theta = 0$. Suppose $\theta(0) = 2$, $\theta'(0) = -6$. What are the period and the amplitude of the resulting oscillations?

Solution. The characteristic equation has roots $\pm 3i$. Hence, the general solution to the DE is $\theta(t) = A\cos(3t) + B\sin(3t)$.

We use the initial conditions to determine A and B: $\theta(0) = A \overset{!}{=} 2$. $\theta'(0) = 3B \overset{!}{=} -6$.

Hence, the unique solution to the IVP is $\theta(t) = 2\cos(3t) - 2\sin(3t)$.

In particular, the period is $2\pi/3$ and the amplitude is $\sqrt{A^2 + B^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8}$.

Problem 6. (2 points) The motion of a certain mass on a spring is described by $y'' + dy' + 5y = 0$ where $d > 0$. For which values of d is the motion underdamped?

Solution. The discriminant of the characteristic equation is $d^2 - 20$. Hence the system is underdamped if $d^2 - 20 < 0$, that is $d < \sqrt{20}$.

Problem 7. (7 points) Fill in the blanks. None of the problems should require any computation!

(a) Write down a homogeneous linear differential equation satisfied by $y(x) = 5 - 2x\sinh(4x) + (3x^2 - 1)e^x$.

Here, and in the next part, you can use the operator D to write the DE. No need to simplify, any form is acceptable.

(b) Let y_p be any solution to the inhomogeneous linear differential equation $y'' - 4y = x^2 - 5e^{2x}$. Find a homogeneous linear differential equation which y_p solves.

(c) Consider a homogeneous linear differential equation with constant real coefficients which has order 4. Suppose $y(x) = 2x - 5e^x\sin(3x)$ is a solution. Write down the general solution.

(d) Name the method which we can use to solve the differential equation $y'' - 4y = \frac{1}{x}$.

Solution.

(a) $D(D-4)^2(D+4)^2(D-1)^3 y = 0$

Explanation. In order for $y(x)$ to be a solution of $p(D)y = 0$, we need to have the characteristic roots $0, \pm 4, \pm 4, 1, 1, 1$. Hence, the simplest DE is $D(D-4)^2(D+4)^2(D-1)^3 y = 0$.

(b) $D^3(D-2)(D^2-4)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2-4)y_p = x^2 - 5e^{2x}$. The inhomogeneous part is a solution of $p(D)y = 0$ if and only if $0,0,0,2$ are roots of the characteristic polynomial $p(D)$. In particular, $D^3(D-2)(x^2-5e^{2x}) = 0$. Combined, we find that $D^3(D-2)(D^2-4)y_p = 0$.

(c) $y(x) = C_1 + C_2 x + C_3 e^x \cos(3x) + C_4 e^x \sin(3x)$.

Explanation. $y(x) = 2x - 5e^x \sin(3x)$ is a solution of $p(D)y = 0$ if and only if $0, 0, 1 \pm 3i$ are roots of the characteristic polynomial $p(D)$. Since the order of the DE is 4, there can be no further roots. Hence, the general solution of this DE is $y(x) = C_1 + C_2 x + C_3 e^x \cos(3x) + C_4 e^x \sin(3x)$.

(d) Variation of constants

Note that we cannot use the method of undetermined coefficients here because the inhomogeneous part $\frac{1}{x}$ is not a solution of a DE of the form $p(D)y = 0$.

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(extra scratch paper)