Midterm #2 – Practice

MATH 238 — Differential Equations I
Midterm: Friday, Apr 11, 2025

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3+7i$ is a repeated characteristic root of L.

(a) What is the general solution to $Ly=0$?

(b) Write down the simplest form of a particular solution y_p of the DE $Ly=7x^2e^{3x}$ with undetermined coefficients.

(c) Write down the simplest form of a particular solution y_p of the DE $Ly=e^{3x}\sin(7x)+3x^2$ with undetermined coefficients.

Problem 2.

(a) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x)=7x-2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.

(b) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x)=2xe^{3x}+x\cos(5x)-5\sin(x)$ is a solution. Write down the general solution.

(c) Write down a homogeneous linear differential equation satisfied by $y(x)=1-5x^2e^{-2x}$.

Here, and elsewhere, you can use the operator D to write the DE. No need to simplify, any form is acceptable.

(d) Write down a homogeneous linear differential equation satisfied by $y(x)=2-3x\sinh(4x)-(7x^2+5)e^x$.

(e) Let y_p be any solution to the inhomogeneous linear differential equation $y''-9y=4xe^x-5e^{2x}$. Find a homogeneous linear differential equation which y_p solves.

Problem 3.

(a) Determine the general solution of the system $\begin{array}{rcl} y_1' &=& y_1-6y_2 \\ y_2' &=& y_1-4y_2 \end{array}$.

(b) Solve the IVP $\begin{array}{rcl} y_1' &=& y_1-6y_2 \\ y_2' &=& y_1-4y_2 \end{array}$ with $\begin{array}{rcl} y_1(0) &=& 4 \\ y_2(0) &=& 1 \end{array}$.

(c) Determine a particular solution to $\begin{array}{rcl} y_1' &=& y_1-6y_2 \\ y_2' &=& y_1-4y_2-2e^{3x} \end{array}$.

(d) Determine the general solution to $\begin{array}{rcl} y_1' &=& y_1-6y_2 \\ y_2' &=& y_1-4y_2-2e^{3x} \end{array}$.

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Problem 4.

(a) Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.

(b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = 4,\ y_2(0) = 0,\ y_2'(0) = -1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y} + \boldsymbol{f},\ \boldsymbol{y}(0) = \boldsymbol{c}$.

Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions.

Problem 6.

(a) What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?

(b) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?

(c) The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?

(d) A forced mechanical oscillator is described by $y'' + 2y' + y = 25\cos(2t)$. As $t \to \infty$, what is the period and the amplitude of the resulting oscillations?

(e) The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5\sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \to \infty$, what are the period and amplitude of the oscillations?

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F(t)$.

(a) Assume first that there is no external force, i.e. $F(t) = 0$. For which values of d is the system overdamped?

(b) Now, $F(t) = \sin(4\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω, if any, does resonance occur?

(c) Now, $F(t) = 5\cos(\omega t) - 2\sin(3\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω, if any, does resonance occur?

Problem 8.

(a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

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(b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.

(c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.

(d) Solve the initial value problem $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Problem 9.

(a) Consider the differential equation $x^2 y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.

(b) Determine the general solution of $x^2 y'' - 4xy' + 6y = x^3$.