

Midterm #2 – Practice

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3 + 7i$ is a repeated characteristic root of L .

- What is the general solution to $Ly = 0$?
- Write down the simplest form of a particular solution y_p of the DE $Ly = 7x^2e^{3x}$ with undetermined coefficients.
- Write down the simplest form of a particular solution y_p of the DE $Ly = e^{3x}\sin(7x) + 3x^2$ with undetermined coefficients.

Solution.

- Since L is real, if $3 + 7i$ is a repeated characteristic root of L , then $3 - 7i$ must be a repeated characteristic root of L as well. Hence, the 4 characteristic roots must be $3 \pm 7i, 3 \pm 7i$.

The corresponding general solution is $(C_1 + C_2x)e^{3x}\cos(7x) + (C_3 + C_4x)e^{3x}\sin(7x)$.

- The characteristic roots of the homogeneous DE are $3 \pm 7i, 3 \pm 7i$ while the roots for the inhomogeneous part are $3, 3, 3$.

Hence, there must a particular solution of the form $y_p = (C_1 + C_2x + C_3x^2)e^{3x}$.

The unique values of C_1, C_2, C_3 for which this is a solution of the DE need to be determined by plugging into the DE.

- The characteristic roots of the homogeneous DE are $3 \pm 7i, 3 \pm 7i$ while the roots for the inhomogeneous part are $3 \pm 7i, 0, 0, 0$.

Hence, there must a particular solution of the form $C_1x^2e^{3x}\cos(7x) + C_2x^2e^{3x}\sin(7x) + C_3 + C_4x + C_5x^2$.

The unique values of C_1, C_2, C_3, C_4, C_5 for which this is a solution of the DE need to be determined by plugging into the DE.

Problem 2.

- Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 7x - 2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.
- Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 2xe^{3x} + x\cos(5x) - 5\sin(x)$ is a solution. Write down the general solution.
- Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2e^{-2x}$.
Here, and elsewhere, you can use the operator D to write the DE. No need to simplify, any form is acceptable.
- Write down a homogeneous linear differential equation satisfied by $y(x) = 2 - 3x\sinh(4x) - (7x^2 + 5)e^x$.
- Let y_p be any solution to the inhomogeneous linear differential equation $y'' - 9y = 4xe^x - 5e^{2x}$. Find a homogeneous linear differential equation which y_p solves.

Solution.

- (a) The characteristic roots must include $0, 0, 3 \pm 5i, 3 \pm 5i, 3 \pm 5i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $C_1 + C_2x + (C_3 + C_4x + C_5x^2)e^{3x}\cos(5x) + (C_6 + C_7x + C_8x^2)e^{3x}\sin(5x)$.

- (b) The characteristic roots must include $3, 3, \pm 5i, \pm 5i, \pm i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $(C_1 + C_2x)e^{3x} + (C_3 + C_4x)\cos(5x) + (C_5 + C_6x)\sin(5x) + C_7\cos(x) + C_8\sin(x)$.

- (c) $y(x) = 1 - 5x^2e^{-2x}$ is a solution of $p(D)y = 0$ if and only if $-2, -2, -2, 0$ are roots of the characteristic polynomial $p(D)$. Hence, the simplest DE is obtained from $p(D) = D(D+2)^3$.

Comment. If we wanted to, we could multiply out $D(D+2)^3 = D^4 + 6D^3 + 12D^2 + 8D$ and write the DE as $y^{(4)} + 6y''' + 12y'' + 8y' = 0$. However, this is usually neither needed nor useful.

- (d) In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, \pm 4, \pm 4, 1, 1, 1$ (note that $\sinh(2x) = \frac{1}{2}(e^{4x} + e^{-4x})$ which contributes the roots ± 4 ; twice because the sinh term is multiplied by x).

Hence, the simplest differential equation is $D(D-4)^2(D+4)^2(D-1)^3y = 0$.

- (e) $(D-1)^2(D-2)(D^2-9)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2-9)y_p = 4xe^x - 5e^{2x}$. The right-hand side $4xe^x - 5e^{2x}$ is a solution of $p(D)y = 0$ if and only if $1, 1, 2$ are roots of the characteristic polynomial $p(D)$. In particular, $(D-1)^2(D-2)(4xe^x - 5e^{2x}) = 0$. Combined, we find that $(D-1)^2(D-2)(D^2-9)y_p = 0$.

Problem 3.

- (a) Determine the general solution of the system
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned}$$

- (b) Solve the IVP
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned} \quad \text{with} \quad \begin{aligned} y_1(0) &= 4 \\ y_2(0) &= 1 \end{aligned}$$

- (c) Determine a particular solution to
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$

- (d) Determine the general solution to
$$\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$$

Solution.

- (a) Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1')$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 0$.

This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-1, -2$.

Hence, $y_1 = C_1e^{-x} + C_2e^{-2x}$.

We can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(C_1e^{-x} + C_2e^{-2x} - (-C_1e^{-x} - 2C_2e^{-2x})) = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

- (b) From the previous part, we know $y_1 = C_1e^{-x} + C_2e^{-2x}$ and $y_2 = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

We solve for C_1 and C_2 using the initial conditions: $y_1(0) = C_1 + C_2 \stackrel{!}{=} 4$ and $y_2(0) = \frac{1}{3}C_1 + \frac{1}{2}C_2 \stackrel{!}{=} 1$.

Solving these two equations, we find $C_1 = 6$ and $C_2 = -2$.

Thus, the unique solution to the IVP is $y_1 = 6e^{-x} - 2e^{-2x}$ and $y_2 = 2e^{-x} - e^{-2x}$.

- (c) We proceed as in the first part to write $y_2 = \frac{1}{6}(y_1 - y_1')$.

Using this in the second equation and simplifying, we get $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are $-1, -2$, while the root for the inhomogeneous part is 3 , there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C . To determine C , we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$.

Having found $y_1 = \frac{3}{5}e^{3x}$, we can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(\frac{3}{5}e^{3x} - \frac{9}{5}e^{3x}) = -\frac{1}{5}e^{3x}$.

- (d) We get the general solution by adding the particular solution (previous part) and the general solution to the corresponding homogeneous equation (first part):

Hence, the general solution is $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$ and $y_2 = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

Alternatively. Here is a solution that proceeds from scratch (rather than referring to previous parts):

Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1') - 2e^{3x}$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are $-1, -2$, while the root for the inhomogeneous part is 3 , there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C . To determine C , we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$ and the particular solution is $y_1 = \frac{3}{5}e^{3x}$. The corresponding general solution is $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$.

We can then determine y_2 as follows:

$$y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(\frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x} - (\frac{9}{5}e^{3x} - C_1e^{-x} - 2C_2e^{-2x})) = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}.$$

Problem 4.

- (a) Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.
- (b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} & y_1(0) &= 1, \quad y_1'(0) = 4, \quad y_2(0) = 0, \quad y_2'(0) = -1 \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned}$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{c}$.

Solution.

- (a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, the DE translates into the first-order system
$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -5y_1 + 4y_2 - 2y_3 + 2\sin(x) \end{cases}.$$

In matrix form, with $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, this is $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 4 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ 2\sin(x) \end{bmatrix}$.

- (b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, with $\mathbf{y} = (y_1, y_2, y_3, y_4)$, the given system translates into

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 7 & -3 & 0 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ e^{2x} \\ -3e^x \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}.$$

Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i , including initial conditions.

Solution. Note that at time t , T_1 contains $10 + 15t - 10t = 10 + 5t$ gal of solution. On the other hand, T_2 contains a constant amount of 10 gal, and T_3 $10 + 10t$ gal of solution.

In the time interval $[t, t + \Delta t]$, we have:

$$\begin{aligned} \Delta y_1 &\approx 15 \cdot 2 \cdot \Delta t - 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t &\implies y_1' &= 30 - \frac{2y_1}{2 + t} \\ \Delta y_2 &\approx 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t - 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_2' &= \frac{2y_1}{2 + t} - y_2 \\ \Delta y_3 &\approx 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_3' &= y_2 \end{aligned}$$

We also have the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$. In matrix form, writing $\mathbf{y} = (y_1, y_2, y_3)$, this is

$$\mathbf{y}' = \begin{bmatrix} -\frac{2}{2+t} & 0 & 0 \\ \frac{2}{2+t} & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients.

Comment. Because of its particularly simple structure, we actually have the techniques to solve this system. Namely, note that the first equation only involves y_1 . It is a linear first-order equation which we could therefore solve using an integrating factor. With y_1 determined, the second differential equation only involves y_2 and is, again, a linear first-order equation. Solving it for y_2 , we then get y_3 by a final integration.

Problem 6.

- (a) What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?
- (b) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?
- (c) The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?
- (d) A forced mechanical oscillator is described by $y'' + 2y' + y = 25 \cos(2t)$. As $t \rightarrow \infty$, what is the period and the amplitude of the resulting oscillations?
- (e) The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5 \sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \rightarrow \infty$, what are the period and amplitude of the oscillations?

Solution.

- (a) The period is $2\pi/7$ and the amplitude is $\sqrt{3^2 + (-5)^2} = \sqrt{34}$.
- (b) The characteristic equation has roots $\pm 2i$. Hence, the general solution to the DE is $\theta(t) = A \cos(2t) + B \sin(2t)$.

We use the initial conditions to determine A and B : $\theta(0) = A = \frac{3}{10}$, $\theta'(0) = 2B = -\frac{4}{5}$.

Hence, the unique solution to the IVP is $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t)$.

In particular, the period is π and the amplitude is $\sqrt{A^2 + B^2} = \sqrt{\frac{9}{100} + \frac{16}{100}} = \frac{1}{2}$.

- (c) The characteristic equation has roots $\frac{1}{2}(-d \pm \sqrt{d^2 - 20})$. The system is underdamped if the solutions involve oscillations, which happens if and only if $d^2 - 20$ (the discriminant) is negative.

Since $d^2 - 20 < 0$ if $d < \sqrt{20}$, the system is underdamped for $d \in (0, \sqrt{20})$.

Correspondingly, the system is critically damped for $d = \sqrt{20}$ and overdamped for $d > \sqrt{20}$.

- (d) The characteristic roots of the homogeneous DE are $-1, -1$ while the roots for the inhomogeneous part are $\pm 2i$. Since they don't overlap, there must be a particular solution y_p of the form $y_p = A \cos(2t) + B \sin(2t)$.

We plug into the DE to find $y_p'' + 2y_p' + y_p = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \stackrel{!}{=} 25\cos(2t)$.

Comparing coefficients, we get $-3A + 4B = 25$ and $-3B - 4A = 0$. Solving these, we find $A = -3$ and $B = 4$.

Hence, $y_p(t) = -3\cos(2t) + 4\sin(2t)$ and the general solution is $y(t) = -3\cos(2t) + 4\sin(2t) + (C_1 + C_2x)e^{-t}$.

As $t \rightarrow \infty$, we have $e^{-t} \rightarrow 0$ so that $y(t) \approx -3\cos(2t) + 4\sin(2t)$.

In particular, the period is π and the amplitude is $\sqrt{(-3)^2 + 4^2} = 5$.

- (e) The characteristic roots of the homogeneous DE are $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$ while the roots for the inhomogeneous part are $\pm i$. Since there is no overlap, there must be a particular solution y_p of form $y_p = A \cos(t) + B \sin(t)$. By plugging into DE, we find $A = -4, B = -2$.

Hence, the general solution is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2))$.

We determine C_1 and C_2 using the initial conditions. From $y(0) = -4 + C_1 \stackrel{!}{=} 2$, we conclude $C_1 = 6$. We then compute $y'(t) = 4\sin(t) - 2\cos(t) - \frac{1}{2}e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2)) + e^{-t/2}(-\frac{1}{2}C_1 \sin(t/2) + \frac{1}{2}C_2 \cos(t/2))$. Hence, $y'(0) = -2 - \frac{1}{2}C_1 + \frac{1}{2}C_2 \stackrel{!}{=} 0$, from which we conclude that $C_2 = 10$.

Therefore, the unique solution to the IVP is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(6 \cos(t/2) + 10 \sin(t/2))$.

For large t , $y(t) \approx -4\cos(t) - 2\sin(t)$ (since $e^{-t/2} \rightarrow 0$). Hence, the period is 2π and the amplitude is $\sqrt{4^2 + 2^2} = \sqrt{20}$.

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F(t)$.

- (a) Assume first that there is no external force, i.e. $F(t) = 0$. For which values of d is the system overdamped?
- (b) Now, $F(t) = \sin(4\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?
- (c) Now, $F(t) = 5\cos(\omega t) - 2\sin(3\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?

Solution.

- (a) The discriminant of the characteristic equation is $d^2 - 24$. Hence the system is overdamped if $d^2 - 24 > 0$, that is $d > \sqrt{24} = 2\sqrt{6}$.
- (b) The natural frequency is $\sqrt{\frac{3}{2}}$. Resonance therefore occurs if $4\omega = \sqrt{\frac{3}{2}}$ or, equivalently, $\omega = \frac{1}{4}\sqrt{\frac{3}{2}}$.
- (c) The natural frequency is $\sqrt{\frac{3}{2}}$. The external frequencies are ω and 3ω . Resonance therefore occurs if $\omega = \sqrt{\frac{3}{2}}$ or $3\omega = \sqrt{\frac{3}{2}}$. Equivalently, resonance occurs if $\omega = \sqrt{\frac{3}{2}}$ or $\omega = \frac{1}{3}\sqrt{\frac{3}{2}} = \frac{1}{\sqrt{6}}$.

Problem 8.

- (a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.
- (b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.
- (c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.
- (d) Solve the initial value problem $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Solution.

- (a) The characteristic equation for the corresponding homogeneous DE has roots 2, 2. The inhomogeneous part (on the right-hand side) solves a DE whose characteristic equation has root 2. Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax^2e^{2x}$.

To determine A , we plug into the DE using $y'_p = 2A(x + x^2)e^{2x}$ and $y''_p = 2A(1 + 4x + 2x^2)e^{2x}$:

$$y''_p - 4y'_p + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \stackrel{!}{=} 3e^{2x}. \text{ Hence, } A = \frac{3}{2} \text{ so that } y_p = \frac{3}{2}x^2e^{2x}.$$

Accordingly, the general solution is $y(x) = (C_1 + C_2x + \frac{3}{2}x^2)e^{2x}$.

- (b) Let us first solve the homogeneous equation $y''' - y = 0$. Its characteristic polynomial $D^3 - 1 = (D - 1)(D^2 + D + 1)$ has roots 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The inhomogeneous part solves a DE whose characteristic equation has roots 0, 1.

Noting the repetition of the root 1, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax e^x + B$.

$$y'_p = A(x + 1)e^x, \quad y''_p = A(x + 2)e^x, \quad y'''_p = A(x + 3)e^x$$

Plugging into the DE, we get $y'''_p - y_p = 3Ae^x - B \stackrel{!}{=} e^x + 7$. Consequently, $A = \frac{1}{3}$, $B = -7$ so that $y_p = -7 + \frac{1}{3}x e^x$.

Hence, the general solution is $y(x) = -7 + (C_1 + \frac{1}{3}x)e^x + C_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Comment. On the exam, you won't be asked to factor a polynomial of degree 3 (except possibly trivial cases like $D^5 - D^3 = D^3(D^2 - 1)$). Here, factoring $D^3 - 1$ is not too bad: since 1 is clearly a root, we know that $D^3 - 1 = (D - 1) \cdot p_2(D)$ where $p_2(D)$ is quadratic polynomial which we can find by long division: $p_2(D) = \frac{D^3 - 1}{D - 1}$.

- (c) Since $D^4 + 6D^3 + 13D^2 = D^2(D^2 + 6D + 13)$, the characteristic equation for the corresponding homogeneous DE has roots 0, 0, $-3 \pm 2i$. The inhomogeneous part solves a DE whose characteristic equation has root 0. Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax^2$.

Plugging into the DE, we get $y_p^{(4)} + 6y_p''' + 13y_p'' = 26A \stackrel{!}{=} 2$. Thus $A = \frac{1}{13}$ so that $y_p = \frac{1}{13}x^2$.

Hence, the general solution is $y(x) = \frac{1}{13}x^2 + C_1 + C_2x + C_3e^{-3x}\cos(2x) + C_4e^{-3x}\sin(2x)$.

- (d) The characteristic equation for the associated homogeneous DE has roots $-1, -1$. The inhomogeneous part solves a DE whose characteristic equation has roots $-1, 2$.

Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ae^{2x} + Bx^2e^{-x}$. To find A, B we plug into the DE. [...] We find $A = \frac{2}{9}$ and $B = \frac{1}{2}$.

Particular solution: $y_p = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x}$

General solution: $y = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + C_1e^{-x} + C_2xe^{-x}$

Now, we use the initial values [...], to find $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$.

Problem 9.

- (a) Consider the differential equation $x^2y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.
- (b) Determine the general solution of $x^2y'' - 4xy' + 6y = x^3$.

Solution.

- (a) Plugging $y(x) = x^r$ into the DE, we get $x^2r(r-1)x^{r-2} - 4xrx^{r-1} + 6x^r = [r(r-1) - 4r + 6]x^r \stackrel{!}{=} 0$.

Since $r(r-1) - 4r + 6 = (r-2)(r-3)$, we find the solutions x^2 and x^3 . Since this is a second-order equation and our solutions are independent, there can be no further solutions.

- (b) We can find a particular solution to this inhomogeneous DE using the method of variation of parameters/constants. From the first part, we know that the corresponding homogeneous DE has the solutions $y_1 = x^2$, $y_2 = x^3$. The Wronskian of these is $W(x) = y_1y_2' - y_1'y_2 = x^4$.

Put the DE in the form $y'' - 4x^{-1}y' + 6x^{-2}y = f(x)$ with $f(x) = x$. Then, by the method of variation of parameters, a particular solution is given by

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx = -x^2 \int 1 dx + x^3 \int \frac{1}{x} dx = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is $y(x) = C_1x^2 + (C_2 + \ln|x|)x^3$.