

# Midterm #1

Please print your name:

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No notes, calculators or tools of any kind are permitted. There are 27 points in total. You need to show work to receive full credit.

**Good luck!**

**Problem 1. (4 points)** Consider the IVP  $\frac{dy}{dx} = 2x - y$  with  $y(1) = 2$ . Approximate the solution  $y(x)$  for  $x \in [1, 2]$  using Euler's method with 2 steps. In particular, what is the approximation of  $y(2)$ ?

**Solution.** The step size is  $h = \frac{2-1}{2} = \frac{1}{2}$ . Since the DE already is in the form  $y' = f(x, y)$ , we apply Euler's method with  $f(x, y) = 2x - y$ :

$$\begin{aligned}x_0 &= 1 & y_0 &= 2 \\x_1 &= \frac{3}{2} & y_1 &= y_0 + hf(x_0, y_0) = 2 + \frac{1}{2} \cdot [2 \cdot 1 - 2] = 2 \\x_2 &= 2 & y_2 &= y_1 + hf(x_1, y_1) = 2 + \frac{1}{2} \cdot \left[ 2 \cdot \frac{3}{2} - 2 \right] = \frac{5}{2}\end{aligned}$$

In particular, the approximation for  $y(2)$  is  $y_2 = \frac{5}{2} = 2.5$ .

**For comparison.** This is a linear DE that we can solve exactly to find  $y(x) = 2(x - 1 + e^{1-x})$ . In particular, for  $y(2)$  we get the exact value  $y(2) = 2\left(1 + \frac{1}{e}\right) \approx 2.736$ .

**Problem 2. (4 points)** Solve the initial value problem  $\frac{dy}{dx} = 4xy^2$  with  $y(2) = 1$ .

**Solution.** The DE is separable:  $y^{-2}dy = 4xdx$ .

Integrating both sides, we find  $-\frac{1}{y} = 2x^2 + C$ . From  $y(2) = 1$ , we conclude that  $C = -9$ .

Thus, the unique solution is  $y = -\frac{1}{2x^2 - 9} = \frac{1}{9 - 2x^2}$ .

**Problem 3. (8 points)** A tank contains 5gal of pure water. It is filled with brine (containing 6lb/gal salt) at a rate of 2gal/min. At the same time, well-mixed solution flows out at a rate of 1gal/min. How much salt is in the tank after  $t$  minutes?

**Solution.** Let  $x(t)$  denote the amount of salt (in lb) in the tank after time  $t$  (in min).

At time  $t$ , the concentration of salt (in lb/gal) in the tank is  $\frac{x(t)}{V(t)}$  where  $V(t) = 5 + (2 - 1)t = 5 + t$  is the volume (in gal) in the tank.

In the time interval  $[t, t + \Delta t]$ :  $\Delta x \approx 2 \cdot 6 \cdot \Delta t - 1 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$ .

Hence,  $x(t)$  solves the IVP  $\frac{dx}{dt} = 12 - \frac{1}{5+t}x$  with  $x(0) = 0$ . Since this is a linear DE, we can solve it as follows:

- We write it in the form  $\frac{dx}{dt} + \frac{1}{5+t}x = 12$ .
- The integrating factor is  $f(t) = \exp\left(\int \frac{1}{5+t} dt\right) = \exp(\ln(5+t)) = 5+t$ .
- Multiply the (rewritten) DE by  $f(t) = 5+t$  to get  $(5+t)\frac{dx}{dt} + x = 12(5+t)$ .  

$$= \frac{d}{dt}[(5+t)x]$$
- Integrate both sides to get  $(5+t)x = 12 \int (5+t)dt = 60t + 6t^2 + C$ .

Hence the general solution to the DE is  $x(t) = \frac{60t + 6t^2 + C}{5+t}$ . Using  $x(0) = 0$ , we find  $\frac{C}{5} = 0$  from which we conclude that  $C = 0$ .

After  $t$  minutes, the tank therefore contains  $x(t) = \frac{60t + 6t^2}{5+t}$  pounds of salt.

(Depending on preference, we can also write  $\frac{60t + 6t^2}{5+t} = 6(5+t) - \frac{150}{5+t}$ .)

**Problem 4. (2 points)** In the differential equation  $(y+5)\frac{dy}{dx} = x - \cos\left(\frac{3y}{x^2+1}\right)$  substitute  $u = \frac{y}{x^2+1}$ .

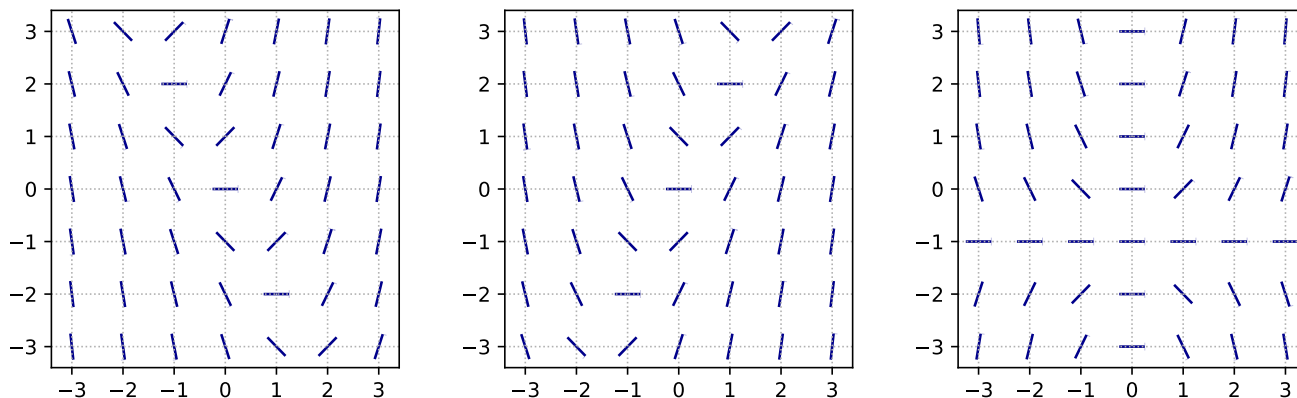
What is the resulting differential equation for  $u$ ?

No need to simplify! Do not attempt to solve!

**Solution.** If  $u = \frac{y}{x^2+1}$ , then  $y = (x^2+1)u$  and  $\frac{dy}{dx} = (x^2+1)\frac{du}{dx} + 2xu$ .

Hence, the resulting differential equation for  $u$  is  $((x^2+1)u+5)\left((x^2+1)\frac{du}{dx} + 2xu\right) = x - \cos(3u)$ .

**Problem 5. (2 points)** Circle the slope field below which belongs to the differential equation  $y' = 2x - y$ .



**Solution.** A good point to carefully consider is  $(1, 2)$ . By the DE, a solution passing through that point has slope  $y'$  satisfying  $y' = 2 \cdot 1 - 2 = 0$ . The only plot compatible with that is the second one.

Of course, we can arrive at the same conclusion based on other points.

[The first plot is for  $y' = 2x + y$ , the second for  $y' = 2x - y$ , and the third for  $y' = x + xy$ .]

**Problem 6. (3 points)** Consider the initial value problem  $(x^2 - 1)y' = 3y^2 + 2$ ,  $y(a) = b$ . For which values of  $a$  and  $b$  can we guarantee existence and uniqueness of a (local) solution?

**Solution.** Let us write  $y' = f(x, y)$  with  $f(x, y) = \frac{3y^2 + 2}{x^2 - 1}$ . Then  $\frac{\partial}{\partial y} f(x, y) = \frac{6y}{x^2 - 1}$ .

Both  $f(x, y)$  and  $\frac{\partial}{\partial y} f(x, y)$  are continuous for all  $(x, y)$  with  $x \neq \pm 1$ .

Hence, if  $a \neq \pm 1$ , then the IVP locally has a unique solution by the existence and uniqueness theorem.

**Problem 7. (4 points)** A rising population is modeled by the equation  $\frac{dP}{dt} = 300P - 2P^2$ . Answer the following questions without solving the differential equation.

- (a) When the population size stabilizes in the long term, how big will the population be?
- (b) What is the population size when it is growing the fastest?

**Solution.**

- (a) Once the population reaches a stable level in the long term, we have  $\frac{dP}{dt} = 0$  (no change in population size).

Hence,  $0 = 300P - 2P^2 = 2P(150 - P)$  which implies that  $P = 0$  or  $P = 150$ . Since the population is rising, it will approach 150 in the long term.

- (b) This is asking when  $\frac{dP}{dt}$  (the population growth) is maximal.

The DE is telling us that this growth is  $f(P) = 300P - 2P^2$ . This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when  $f'(P) = 0$ .

$$f'(P) = 300 - 4P = 0 \text{ leads to } P = 75.$$

Thus, the population is growing the fastest when its size is 75.

(extra scratch paper)