Example 150. (review) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2-s-6=(s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A=-\frac{6s-23}{s+2}\bigg|_{s=3}=1, \quad B=-\frac{6s-23}{s-3}\bigg|_{s=-2}=-7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}-\frac{7}{s+2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)-7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right)=e^{3t}-7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2}=A+\frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 139 where we considered the same functions.

Example 151. Consider the IVP $y''-3y'+y=2e^{5t}$, $y(0)=-1$, $y'(0)=4$.
Determine the Laplace transform of the unique solution.

Solution. The DE $y''-3y'+y=2e^{5t}$ (plus initial conditions!) transforms into

$$s^2Y-sy(0)-y'(0)-3(sY-y(0))+Y=(s^2-3s+1)Y+(s-7)=\frac{2}{s-5}.$$

Accordingly, $Y(s)=\frac{1}{s^2-3s+1}\left[\frac{2}{s-5}-s+7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3\pm\sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.

Handling discontinuities with the Laplace transform

Let $u_a(t)=\begin{cases} 1, & \text{if } t\geqslant a, \\ 0, & \text{if } t<a, \end{cases}$ be the **unit step function**. Throughout, we assume that $a\geqslant 0$.

Comment. The special case $u_0(t)$ is also known as the **Heaviside function**, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t)=u_0(t-a)$.

Example 152. If $a<b$, then $u_a(t)-u_b(t)=\begin{cases} 1, & \text{if } a\leqslant t<b, \\ 0, & \text{otherwise.} \end{cases}$

Comment. See Example 154 for how to write piecewise-defined functions as combinations of unit step functions.

The following is a useful addition to our table of Laplace transforms:

Example 153. (new entry) We add the following to our table of Laplace transforms:

$$\begin{aligned}\mathcal{L}(u_a(t)f(t-a)) &= \int_a^\infty e^{-st}f(t-a)\mathrm{d}t=\int_0^\infty e^{-s(\tilde{t}+a)}f(\tilde{t})\mathrm{d}\tilde{t} \\ &= e^{-as}\int_0^\infty e^{-s\tilde{t}}f(\tilde{t})\mathrm{d}\tilde{t}=e^{-as}F(s)\end{aligned}$$

Comment. Note that the graph of $u_a(t)f(t-a)$ is the same as $f(t)$ but delayed by a (make a sketch!).

In particular. $\mathcal{L}(u_a(t))=\frac{e^{-sa}}{s}$

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The next example illustrates that any piecewise defined function can be written using a single formula involving step functions. This is based on the simple observation from Example 152 that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Comment. For our present purposes, we don't really care about the precise value of a function at a single point. Specifically, it doesn't really matter which value the function $u_a(t) - u_b(t)$ takes at $t = b$ (technically, the value is 0 but it may as well be 1 since there is a discontinuity at $t = b$).

Example 154. Write $f(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2, & \text{if } 0 \leqslant t < 1, \\ 3, & \text{if } 1 \leqslant t < 2, \\ \cos(t-2), & \text{if } t \geqslant 2, \end{cases}$ as a combination of unit step functions.

Solution. $f(t) = t^2(u_0(t) - u_1(t)) + 3(u_1(t) - u_2(t)) + \cos(t-2)u_2(t)$

Homework. Compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t) f(t-a)) = e^{-sa} F(s)$ with $a = 1$ and $f(t-1) = t^2$. Then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$. Combined, we get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.