Notes for Lecture 31

The inverse Laplace transform

Theorem 142. (uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2(t)$.

Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

We say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

Advanced comment. This uniqueness is true for continuous functions f_1, f_2. It is also true for piecewise continuous functions except at those values of t for which there is a discontinuity. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Example 143. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{5}{s+3}\right)$.

Solution. In other words, if $F(s) = \frac{5}{s+3}$, what is $f(t)$?

$\mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 5e^{-3t}$

Example 144. (extra) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s-7}{s^2+4}\right)$.

Solution. In other words, if $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?

$F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$.

Example 145. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2 - s - 6 = (s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A = -\frac{6s-23}{s+2}\bigg|_{s=3} = 1, \quad B = -\frac{6s-23}{s-3}\bigg|_{s=-2} = -7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{7}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - 7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right) = e^{3t} - 7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2} = A + \frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 139 where we considered the same functions.

Example 146. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)$.

Solution. Note that $s^2 - s - 2 = (s-2)(s+1)$. We use partial fractions to write $\frac{s+13}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$. We find the coefficients as

$$A = \frac{s+13}{s+1}\bigg|_{s=2} = 5, \quad B = \frac{s+13}{s-2}\bigg|_{s=-1} = -4.$$

Hence $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right) = \mathcal{L}^{-1}\left(\frac{5}{s-2} - \frac{4}{s+1}\right) = 5e^{2t} - 4e^{-t}$.

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Solving simple DEs using the Laplace transform

In the following examples, we write $Y(s)$ for the Laplace transform of $y(t)$.

Recall from our Laplace transform table that this implies that the Laplace transform of $y'(t)$ is $sY(s) - y(0)$.

Example 147. Solve the (very simple) IVP $y'(t) - 2y(t) = 0$, $y(0) = 7$.

At this point, you might be able to "see" right away that the unique solution is $y(t) = 7e^{2t}$.

Solution. (old style) The characteristic root is 2, so that the general solution is $y(t) = Ce^{2t}$. Using the initial condition, we find that $C = 7$, so that $y(t) = 7e^{2t}$.

Solution. (Laplace style) $y' - 2y = 0$ transforms into

$$\mathcal{L}(y'(t) - 2y(t)) = \mathcal{L}(y'(t)) - 2\mathcal{L}(y(t)) = sY(s) - y(0) - 2Y(s) = (s-2)Y(s) - 7 = 0.$$

This is an algebraic equation for $Y(s)$. It follows that $Y(s) = \frac{7}{s-2}$. By inverting the Laplace transform, we conclude that $y(t) = 7e^{2t}$.

Example 148. Solve the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$ has roots $1, 2$.

The characteristic root for the inhomogeneous part is -1. Since there is no duplication, there must be a particular solution of the form $y_p(t) = Ae^{-t}$.

To determine A, we plug into the DE $y_p'' - 3y_p' + 2y_p = 6Ae^{-t} \overset{!}{=} e^{-t}$ and conclude $A = \frac{1}{6}$.

The general solution thus is $y(t) = \frac{1}{6}e^{-t} + C_1e^t + C_2e^{2t}$. We need to find C_1 and C_2 using the initial conditions.

Solving $y(0) = \frac{1}{6} + C_1 + C_2 \overset{!}{=} 0$ and $y'(0) = -\frac{1}{6} + C_1 + 2C_2 \overset{!}{=} 1$, we find $C_2 = \frac{4}{3}$ and $C_1 = -\frac{3}{2}$.

Hence, the unique solution to the IVP is $y(t) = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$.

Solution. (Laplace style) The differential equation (plus initial conditions!) transforms as follows:

$$\begin{aligned}
\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^{-t}) \\
s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s+1} \\
(s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s+1} = \frac{s+2}{s+1} \\
Y(s) &= \frac{s+2}{(s^2 - 3s + 2)(s+1)} \\
&= \frac{s+2}{(s-1)(s-2)(s+1)}
\end{aligned}$$

To find $y(t)$, we use partial fractions to write $Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$. We find the coefficients as

$$A = \left.\frac{s+2}{(s-2)(s+1)}\right|_{s=1} = -\frac{3}{2}, \quad B = \left.\frac{s+2}{(s-1)(s+1)}\right|_{s=2} = \frac{4}{3}, \quad C = \left.\frac{s+2}{(s-1)(s-2)}\right|_{s=-1} = \frac{1}{6}.$$

Hence, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}\right) = Ae^t + Be^{2t} + Ce^{-t} = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$, as above.

Comment. Note the factor $s^2 - 3s + 2$ in front of $Y(s)$ when we transformed the DE. This is the characteristic polynomial. Can you see how the characteristic roots of the homogeneous DE and the inhomogeneous part show up in the Laplace transform approach?

Example 149. Consider the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

Determine the Laplace transform of the unique solution.

Solution. We just did that! By transforming the DE, we found that $Y(s) = \frac{s+2}{(s-1)(s-2)(s+1)}$.