Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y = 0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j = 0, 1, \ldots, k-1$.

Example 77. (review) Find the general solution of $y''' + 2y'' + y' = 0$.

Solution. The characteristic polynomial $p(D) = D(D+1)^2$ has roots $0, 1, 1$.
Hence, the general solution is $A + (B + Cx)e^x$.

Example 78. Determine the general solution of $y''' - 3y'' + 3y' - y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D^2 + 3D - 1 = (D-1)^3$ has roots $1, 1, 1$.
By Theorem 74, the general solution is $y(x) = (C_1 + C_2 x + C_3 x^2)e^x$.

Comment. The coefficients $1, 2, 1$ and $1, 3, 3, 1$ in $(D+1)^2$ and $(D+1)^3$ are known as binomial coefficients. They can be arranged as rows in Pascal's triangle where the next row would be $1, 4, 6, 4, 1$.

Example 79. Determine the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.
Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 80. Determine the general solution of $y^{(6)} = 3y^{(5)} - 4y'''$.

Solution. This DE is of the form $p(D)\, y = 0$ with $p(D) = D^6 - 3D^5 + 4D^3 = D^3(D-2)^2(D+1)$.
The characteristic roots are $2, 2, 0, 0, 0, -1$.
By Theorem 74, the general solution is $y(x) = (C_1 + C_2 x)e^{2x} + C_3 + C_4 x + C_5 x^2 + C_6 e^{-x}$.

Example 81. Consider the function $y(x) = 3x e^{-2x} + 7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $-2, -2, 1$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2(D-1)$.

Note. For many purposes it is best to leave the DE as $(D+2)^2(D-1)y = 0$. On the other hand, if we wanted to, we could multiply out $(D+2)^2(D-1) = D^3 + 3D^2 - 4$ to write the DE in the more classical form $y''' + 3y'' - 4y = 0$.

Example 82. Consider the function $y(x) = 3x e^{-2x} + 7$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $-2, -2, 0$.
The simplest choice for $p(D)$ thus is $p(D) = (D+2)^2 D$.

Note. Optionally, we can expand $(D+2)^2 D = D^3 + 4D^2 + 4D$ to write the DE as $y''' + 4y'' + 4y' = 0$.

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Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\mathrm{Re}(z) = x$.

  Likewise the **imaginary part** is $\mathrm{Im}(z) = y$.

  Observe that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 83. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Comment. It follows that $\cos(x) = \mathrm{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \mathrm{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Example 84. Determine the general solution of $y'' + y = 0$.

Solution. (complex numbers in general solution) The characteristic polynomial is $D^2 + 1$ which has no roots over the reals. Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. (real general solution) On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions. Hence, the general solution can also be written as $y(x) = D_1\cos(x) + D_2\sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (Theorem 83) by which $e^{\pm ix} = \cos(x) \pm i\sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 85. Determine the general solution of $y'' - 4y' + 13y = 0$ using only real numbers.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

[We can use the quadratic formula to find these roots as $\frac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$.]

Hence, the general solution in real form is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.

Note. $e^{(2 \pm 3i)x} = e^{2x}e^{\pm 3ix} = e^{2x}(\cos(3x) \pm i\sin(3x))$

Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The **general solution of linear DE** always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any solution (called a **particular solution**) and y_1, y_2, ..., y_n are solutions to the corresponding **homogeneous** linear DE.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x)$.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE. The corresponding **homogeneous** DE is $Ly = 0$ (note that the zero function $y(x) = 0$ is a solution of $Ly = 0$).
- L is called a **linear differential operator**.
  - $L(C_1 y_1 + C_1 y_2) = C_1 L y_1 + C_2 L y_2$ **(linearity)**
    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
  - So, if y_1 solves $Ly = f(x)$, and y_2 solves $Ly = g(x)$, then $C_1 y_1 + C_2 y_2$ solves the differential equation $Ly = C_1 f(x) + C_2\, g(x)$.
  - In particular, if y_1 and y_2 solve the homogeneous DE (then $f(x) = 0$ and $g(x) = 0$), then so does any linear combination $C_1 y_1 + C_2 y_2$. This explains why, for any homogeneous linear DE of order n, there are n solutions y_1, y_2, ..., y_n such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n\, y_n(x)$. Moreover, in that case, if we have a **particular solution** y_p of the inhomogeneous DE $Ly = f(x)$, then $y_p + C_1 y_1 + \ldots + C_n\, y_n$ is the general solution of $Ly = f(x)$.

Example 86. (preview) Determine the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. Because of the hint, we know that a particular solution is $y_p = 3x$.
The homogeneous DE $p(D) y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]
Therefore, the general solution to the original DE is $y_p + C_1\, y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Just to make sure. The DE in operator notation is $Ly = f(x)$ with $L = D^2 + 4$ and $f(x) = 12x$.
Next. How to find the particular solution $y_p = 3x$ ourselves.

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