Review. Every homogeneous linear DE with constant coefficients can be written as $p(D)y = 0$, where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and $p(D)$ is the characteristic polynomial polynomial. Each root r of the characteristic polynomial gives us one solution, namely $y = e^{rx}$, of the DE.

Example 71.

(a) Determine the general solution of $y''' + 7y'' + 14y' + 8y = 0$.

(b) Determine the general solution of $y^{(4)} + 7y''' + 14y'' + 8y' = 0$.

Solution.

(a) This DE is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D^3 + 7D^2 + 14D + 8$.
The characteristic polynomial factors as $p(D) = (D+1)(D+2)(D+4)$.
Hence, we found the solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $y_3 = e^{-4x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$ (see Theorem 64).

(b) The DE now is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D(D^3 + 7D^2 + 14D + 8)$.
Hence, the characteristic polynomial factors as $p(D) = D(D+1)(D+2)(D+4)$ and we find the additional solution $y_4 = e^{0x} = 1$. Thus, the general solution is $y(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$.
Comment. If we didn't know about roots of characteristic polynomials, an alternative approach would be to substitute $u = y'$, resulting in the DE $u''' + 7u'' + 14u' + 8u = 0$. From the first part, we know that $u(x) = C_1\,e^{-x} + C_2\,e^{-2x} + C_3\,e^{-4x}$. Hence, $y(x) = \int u(x)\mathrm{d}x = -C_1\,e^{-x} - \frac{1}{2}C_2\,e^{-2x} - \frac{1}{4}C_3\,e^{-4x} + C$.
Make sure you see that this is an equivalent way of presenting the general solution! (For instance, since C_3 can be any constant, it doesn't make a difference whether we write $-\frac{1}{4}C_3$ or C_3. The latter is preferable unless the $-\frac{1}{4}$ is useful for some purpose.)

Example 72. Determine the general solution of $y''' - y'' - 4y' + 4y = 0$.

Solution. This DE is of the form $p(D)\,y = 0$ with characteristic polynomial $p(D) = D^3 - D^2 - 4D + 4$.
The characteristic polynomial factors as $p(D) = (D-1)(D-2)(D+2)$.
Hence, we found the solutions $y_1 = e^{x}$, $y_2 = e^{2x}$, $y_3 = e^{-2x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x) = C_1\,e^{x} + C_2\,e^{2x} + C_3\,e^{-2x}$.

In this manner, we are able to solve any homogeneous linear DE of order n with constant coefficients provided that there are n different roots r (each giving rise to one solution e^{rx}).

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following example suggests how to get our hands on the missing solutions.

Example 73. Determine the general solution of $y''' = 0$.

Solution. We know from Calculus that the general solution is $y(x) = C_1 + C_2\,x + C_3\,x^2$.

Solution. (looking ahead) The characteristic polynomial $p(D) = D^3$ has roots $0, 0, 0$. By Theorem 74 below, we have the solutions $y(x) = x^j\,e^{0x} = x^j$ for $j = 0, 1, 2$, so that the general solution is $y(x) = C_1 + C_2\,x + C_3\,x^2$.

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Theorem 74. Consider the homogeneous linear DE with constant coefficients $p(D)y=0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r=a\pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j=0,1,\ldots,k-1$. This case will be discussed next time.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D)=q(D)\,(D-r)^k$.
We need to find k solutions to the simpler DE $(D-r)^k y=0$.
It is natural to look for solutions of the form $y=c(x)e^{rx}$.
[This idea is called **variation of constants** since we know that this is a solution if $c(x)$ is a constant.]
Note that $(D-r)[c(x)e^{rx}]=(c'(x)e^{rx}+c(x)re^{rx})-rc(x)e^{rx}=c'(x)e^{rx}$.
Repeating, we get $(D-r)^2[c(x)e^{rx}]=(D-r)[c'(x)e^{rx}]=c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}]=c^{(k)}(x)e^{rx}$.
In particular, $(D-r)^k y=0$ is solved by $y=c(x)e^{rx}$ if and only if $c^{(k)}(x)=0$.
The DE $c^{(k)}(x)=0$ is clearly solved by x^j for $j=0,1,\ldots,k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 75. Determine the general solution of $y'''-3y'+2y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D+2=(D-1)^2(D+2)$ has roots $1,1,-2$.
By Theorem 74, the general solution is $y(x)=(C_1+C_2x)e^x+C_2e^{-2x}$.

Example 76. **(homework)** Solve the following initial value problems:

(a) $y'''=4y''-4y'$ with $y(0)=4$, $y'(0)=0$, $y''(0)=-4$.

(b) $y'''=8y''-16y'$ with $y(0)=1$, $y'(0)=4$, $y''(0)=0$.

Solution.

(a) The characteristic polynomial $p(D)=D^3-4D^2+4D=D(D-2)^2$ has roots $0,2,2$.
By Theorem 74, the general solution is $y(x)=C_1+(C_2+C_3x)e^{2x}$.
From this formula for $y(x)$, we compute $y'(x)=(2C_2+C_3+2C_3x)e^{2x}$ and $y''(x)=4(C_2+C_3+C_3x)e^{2x}$. The initial conditions therefore result in the equations $C_1+C_2=4$, $2C_2+C_3=0$, $4C_2+4C_3=-4$.
Solving these (start with the last two equations) we find $C_1=3$, $C_2=1$, $C_3=-2$.
Hence the unique solution to the IVP is $y(x)=3+(1-2x)e^{2x}$.

(b) The characteristic polynomial $p(D)=D^3-8D^2+16D=D(D-4)^2$ has roots $0,4,4$.
By Theorem 74, the general solution is $y(x)=C_1+(C_2+C_3x)e^{4x}$.
Using $y'(x)=(4C_2+C_3+4C_3x)e^{4x}$ and $y''(x)=4(4C_2+2C_3+4C_3x)e^{4x}$, the initial conditions result in the equations $C_1+C_2=1$, $4C_2+C_3=4$, $16C_2+8C_3=0$.
Solving these (start with the last two equations) we find $C_1=-1$, $C_2=2$, $C_3=-4$.
Hence the unique solution to the IVP is $y(x)=-1+(2-4x)e^{4x}$.