Review. We saw in Theorem 64 that, for a linear DE of order n, the general solution is of the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n y_n(x),$$

where y_p is any single solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

Linear differential equations with constant coefficients

Let us have another look at Example 10. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 67. (warmup) Find the general solution to $y'' = y' + 6y$.

Solution. As in Example 10, we look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6 e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.
This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).
This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.
Since this a homogeneous linear DE (see Theorem 64), the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Homogeneous linear DEs with constant coefficients

Let us look at another example like Example 67. This time we also take an operator approach that explains further what is going on (and that will be particularly useful when we discuss inhomogeneous equations).

An **operator** takes a function as input and returns a function as output. That is exactly what the derivative does.

In the sequel, we write $D = \frac{\mathrm{d}}{\mathrm{d}x}$ for the derivative operator.

For instance. We write $y' = \frac{\mathrm{d}}{\mathrm{d}x} y = Dy$ as well as $y'' = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y = D^2 y$.

Example 68. Find the general solution to $y'' - y' - 2y = 0$.

Solution. (our earlier approach) Let us look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2 e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is the characteristic equation. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE (see Theorem 64), the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operator approach) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2) y = 0$.
Note that $D^2 - D - 2 = (D - 2)(D + 1)$ is the **characteristic polynomial**.
Observe that we get solutions to $(D - 2)(D + 1) y = 0$ from $(D - 2) y = 0$ and $(D + 1) y = 0$.
$(D - 2) y = 0$ is solved by $y_1 = e^{2x}$, and $(D + 1) y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.
Again, we conclude that the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D) y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

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Example 69. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From Example 68, we know that the general solution is $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
It follows that $y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$. We now use the two initial conditions to solve for C_1 and C_2:
The initial conditions result in the two equations $y(0) = C_1 + C_2 \overset{!}{=} 4$, $y'(0) = 2C_1 - C_2 \overset{!}{=} 5$.
Solving these we find $C_1 = 3$, $C_2 = 1$.
Hence the unique solution to the IVP is $y(x) = 3e^{2x} + e^{-x}$.

Example 70. (preview of inhomogeneous linear DEs)

(a) Check that $y = -3x$ is a solution to $y'' - y' - 2y = 6x + 3$.

Comment. We will soon learn how to find such a solution from scratch.

(b) Using the first part, determine the general solution to $y'' - y' - 2y = 6x + 3$.

(c) Determine $f(x)$ so that $y = 7x^2$ solves $y'' - y' - 2y = f(x)$.

Comment. This is how you can create problems like the ones in the first two parts.

Solution.

(a) If $y = -3x$, then $y' = -3$ and $y'' = 0$. Plugging into the DE, we find $0 - (-3) - 2 \cdot (-3x) = 6x + 3$, which verifies that this is a solution.

(b) This is an inhomogeneous linear DE. From Example 68, we know that the corresponding homogeneous DE has the general solution $C_1 e^{2x} + C_2 e^{-x}$.
From the first part, we know that $-3x$ is a particular solution.
Combining this, the general solution to the present DE is $-3x + C_1 e^{2x} + C_2 e^{-x}$ (see Theorem 64).

(c) If $y = 7x^2$, then $y' = 14x$ and $y'' = 14$ so that $y'' - y' - 2y = 14 - 14x - 14x^2$.
Thus $f(x) = 14 - 14x - 14x^2$.

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