Application: Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already-falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the gravitational acceleration is $g \approx 9.81 \text{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modeled by

$$y''(t) = -g.$$

Example 62. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t)\mathrm{d}t = -\frac{1}{2}gt^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2}gt^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2}gt^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.52)| \approx 44.3\,\mathrm{m}/s$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

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Example 63. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\dfrac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.

Linear DEs of higher order

The most general linear first-order DE is of the form $A(x) y' + B(x) y + C(x) = 0$. Any such DE can be rewritten in the form $y' + P(x) y = f(x)$ by dividing by $A(x)$ and rearranging.

We have learned how to solve all of these using an integrating factor.

Linear DEs of order n are those that can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = 0,$$

and it plays an important role in solving the original linear DE.

Important note. A linear DE is **homogeneous** if and only if the zero function $y(x) = 0$ is a solution.

Advanced comment. As we observed in the first-order case, if I is an interval on which all the $P_j(x)$ as well as $f(x)$ are continuous, then for any $a \in I$ the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always has a unique solution (which is defined on all of I).

Theorem 64. (general solution of linear DEs) For a linear DE of order n, the general solution always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n y_n(x),$$

where y_p is any single solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

Comment. If the linear DE is already homogeneous, then the zero function $y(x) = 0$ is a solution and we can use $y_p = 0$. In that case, the general solution is of the form $y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$.

Why? This structure of the solution follows from the observation in the next example.

Example 65. Suppose that y_1 solves $y'' + P(x)y' + Q(x)y = f(x)$ and that y_2 solves $y'' + P(x)y' + Q(x)y = g(x)$ (note that the corresponding homogeneous DE is the same).
Show that $7y_1 + 4y_2$ solves $y'' + P(x)y' + Q(x)y = 7f(x) + 4g(x)$.

Solution. $(7y_1 + 4y_2)'' + P(x)(7y_1 + 4y_2)' + Q(x)(7y_1 + 4y_2)$
$= 7\{y_1'' + P(x)y_1' + Q(x)y_1\} + 4\{y_2'' + P(x)y_2' + Q(x)y_2\} = 7 \cdot f(x) + 4 \cdot g(x)$

Comment. Of course, there is nothing special about the coefficients 7 and 4.

Important comment. In particular, if both $f(x)$ and $g(x)$ are zero, then $7f(x) + 4g(x)$ is zero as well. This shows that homogeneous linear DEs have the important property that, if y_1 and y_2 are two solutions, then any linear combination $C_1 y_1 + C_2 y_2$ is a solution as well.

The upshot is that this observation reduces the task of finding the general solution of a homogeneous linear DE to the task of finding n (sufficiently) different solutions.

Example 66. (extra) The DE $x^2y'' + 2xy' - 6y = 0$ has solutions $y_1 = x^2$, $y_2 = x^{-3}$.

(a) Determine the general solution.

(b) Solve the IVP $x^2y'' + 2xy' - 6y = 0$ with $y(2) = 10$, $y'(2) = 15$.

Solution.

(a) Note that this is a homogeneous linear DE of order 2.
Hence, given the two solutions, we conclude that the general solution is $y(x) = Ax^2 + Bx^{-3}$ (in this case, the particular solution is $y_p = 0$ because the DE is homogeneous).

(b) We already know that the general solution of the DE is $y(x) = Ax^2 + Bx^{-3}$.
It follows that $y'(x) = 2Ax - 3Bx^{-4}$.
We now use the two initial conditions to solve for A and B:
Solving $y(2) = 4A + B/8 \stackrel{!}{=} 10$ and $y'(2) = 4A - 3/16B \stackrel{!}{=} 15$ for A and B results in $A = 3$, $B = -16$.
Hence, the unique solution to the IVP is $y(x) = 3x^2 - 16/x^3$.