Net ID:	Name:

Midterm #3

MATH 286 — Differential Equations Plus
Thursday, April 17

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

Problem 1. (10 points) Let A be a 5×5 matrix with eigenvalues $\pm 3i, 1, 1, 1$.

(a) Suppose that the eigenvalue $\lambda = 1$ has defect 1. Does the equation $\boldsymbol{x}' = A\boldsymbol{x}$ have (nonzero) solutions of one of the following forms?

$$(\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^t \qquad \left(\boldsymbol{v}_1 \frac{t^2}{2} + \boldsymbol{v}_2 t + \boldsymbol{v}_3\right) e^t \qquad \left(\boldsymbol{v}_1 \frac{t^3}{6} + \boldsymbol{v}_2 \frac{t^2}{2} + \boldsymbol{v}_3 t + \boldsymbol{v}_4\right) e^t \qquad (\boldsymbol{v}_1 t + \boldsymbol{v}_2) \sin(3t) \qquad \boldsymbol{v}_1 e^t \cos(3t)$$

Circle those that are solutions (for appropriate choices of the coefficients $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$).

(b) Now, consider the differential equation $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 3t^2, & 0, & \cos(t), & 0, & -1 \end{pmatrix}^T$. Write down a particular solution $\boldsymbol{x}_p$ with undetermined coefficients.

Solution.

(a) $(\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^t$ is the only form, among the ones listed, of which there exists a nonzero solution.

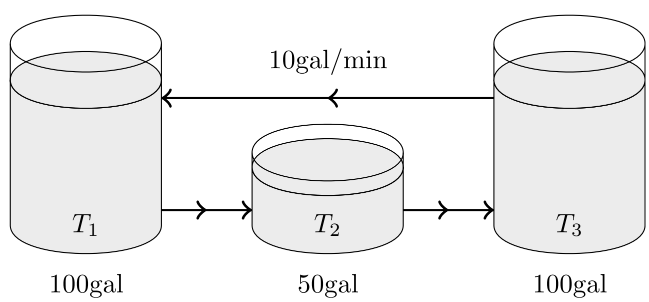
(b) $\boldsymbol{x}_p = \boldsymbol{a}_1 t^2 + \boldsymbol{a}_2 t + \boldsymbol{a}_3 + \boldsymbol{a}_4 \cos(t) + \boldsymbol{a}_5 \sin(t)$. □

Problem 2. (10 points) Three brine tanks T_1, T_2, T_3 are connected as indicated in the sketch below.

The mixtures in each tank are kept uniform by stirring. Suppose that the mixture circulates between the tanks at the rate of 10gal/min. T_1 and T_3 contain 100gal of brine and T_2 contains 50gal.

Denote by $x_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the x_i.

(Do *not* solve the system.)



Solution. In the time interval $[t, t + \Delta t]$, we have:

$$\begin{aligned}
\Delta x_1 &\approx 10 \cdot \frac{x_3}{100} \cdot \Delta t - 10 \cdot \frac{x_1}{100} \cdot \Delta t &\quad\Longrightarrow\quad x_1' &= \frac{1}{10} x_3 - \frac{1}{10} x_1 \\
\Delta x_2 &\approx 10 \cdot \frac{x_1}{100} \cdot \Delta t - 10 \cdot \frac{x_2}{50} \cdot \Delta t &\quad\Longrightarrow\quad x_2' &= \frac{1}{10} x_1 - \frac{1}{5} x_2 \\
\Delta x_3 &\approx 10 \cdot \frac{x_2}{50} \cdot \Delta t - 10 \cdot \frac{x_3}{100} \cdot \Delta t &\quad\Longrightarrow\quad x_3' &= \frac{1}{5} x_2 - \frac{1}{10} x_3
\end{aligned}$$

Optional: in matrix form, writing $\boldsymbol{x} = (x_1, x_2, x_3)$, this is

$$\boldsymbol{x}' = \begin{pmatrix} -\frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{10} \end{pmatrix} \boldsymbol{x}.$$

□

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Problem 3. (20 points) Let $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$.

(a) Find two linearly independent solutions to the linear system $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$.

(b) Compute e^{tA}.

Solution.

(a) The characteristic equation is $(1-\lambda)(-7-\lambda)+16 = \lambda^2 + 6\,\lambda + 9 = 0$. So $\lambda = -3$ is an eigenvalue of A with multiplicity 2. If $\boldsymbol{v} = (a, b)^T$ is an eigenvector, then $a = b$, so the defect of λ is 1 and we must build a length 2 chain $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of generalized eigenvectors. Taking $\boldsymbol{v}_1 = (1, 1)^T$, one then gets $\boldsymbol{v}_1 = (A + 3I)\,\boldsymbol{v}_2$, which has $\boldsymbol{v}_2 = (1/4, 0)^T$ as a solution. We therefore obtain the following two linearly independent solutions:

$$\boldsymbol{x}_1(t) = e^{-3t}\,\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} \quad \text{and} \quad \boldsymbol{x}_2(t) = e^{-3t}\,(t\,\boldsymbol{v}_1 + \boldsymbol{v}_2) = \begin{pmatrix} t + \frac{1}{4} \\ t \end{pmatrix} e^{-3t}.$$

(b) From the first part, we get the fundamental matrix

$$\Phi(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)) = e^{-3t}\begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^{tA} &= \Phi(t)\,\Phi(0)^{-1} = \begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix} e^{-3t} \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{-1} \\ &= e^{-3t}\begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} 1 + 4t & -4t \\ 4t & 1 - 4t \end{pmatrix} = e^{-3t}\begin{pmatrix} 1 + 4t & -4t \\ 4t & 1 - 4t \end{pmatrix}. \end{aligned}$$

□

Problem 4. (20 points) Let A be a 3×3 matrix such that $e^{tA} = \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) What are the eigenvalues of A and what are their defects?

(b) Solve the initial value problem $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$, $\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(c) Find a particular solution to the inhomogeneous linear system $\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix}$.

(d) Find the matrix A.

Solution.

(a) After examining the columns of e^{tA} (which are linearly independent solutions to the given DE) and noting that any solution to the DE is a linear combination of eigenvalue solutions, we see that the only eigenvalue of A is $\lambda = 0$ with multiplicity 3. The defect is 2 since terms of the form $t\,e^{\lambda t}, t^2 e^{\lambda t}$ appear in the columns of e^{tA}.

(b) The initial value problem is solved by

$$\boldsymbol{x}(t) = e^{tA}\,\boldsymbol{x}(0) = \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -t-t^2 \\ t-t^2 \\ 1 \end{pmatrix}.$$

(c) By variation of constants,

$$\begin{aligned}
\boldsymbol{x}_p(t) &= e^{At}\int e^{-At}\begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix}\mathrm{d}t = e^{At}\int \begin{pmatrix} 1-t & t & t-t^2 \\ -t & 1+t & -t+t^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix}\mathrm{d}t \\
&= e^{At}\int \begin{pmatrix} 2/t^2 \\ 2/t^3 + 2/t^2 \\ 0 \end{pmatrix}\mathrm{d}t \\
&= \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2/t \\ -1/t^2 - 2/t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -2/t - 2 + 1/t + 2 \\ -2 - 1/t^2 - 2/t + 1/t + 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/t \\ -1/t^2 - 1/t \\ 0 \end{pmatrix}
\end{aligned}$$

is a particular solution.

(d) We find A as

$$A = \left[\frac{\mathrm{d}}{\mathrm{d}t}e^{tA}\right]_{t=0} = \left[\begin{pmatrix} 1 & -1 & -1-2t \\ 1 & -1 & 1-2t \\ 0 & 0 & 0 \end{pmatrix}\right]_{t=0} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Problem 5. (15 points) Find four independent real-valued solutions of

$$\boldsymbol{x}' = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \boldsymbol{x}.$$

You may use that the characteristic polynomial has the repeated roots $3 \pm 4i$. Moreover, you may use that

$$\boldsymbol{v}_2 = (\ 0\ \ 0\ \ 1\ \ i\ )^T$$

is a generalized eigenvector of rank 2 for $\lambda = 3 - 4i$.

Solution. We first find the corresponding eigenvector $\boldsymbol{v}_1$ as

$$\boldsymbol{v}_1 = (A - \lambda I)\boldsymbol{v}_2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}.$$

The chain induces the two solutions

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{v}_1 e^{(3-4i)t} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} (\cos(4t) - i \sin(4t)) e^{3t} = \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + i \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} \\ \boldsymbol{x}_2 &= (\boldsymbol{v}_1 t + \boldsymbol{v}_2)\, e^{(3-4i)t} = \begin{pmatrix} t \\ it \\ 1 \\ i \end{pmatrix} (\cos(4t) - i \sin(4t)) e^{3t} = \begin{pmatrix} \cos(4t)\, t \\ \sin(4t)\, t \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} + i \begin{pmatrix} -\sin(4t)\, t \\ \cos(4t)\, t \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}. \end{aligned}$$

Taking real and imaginary part, this gives the four real-valued solutions:

$$\begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}, \quad \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}, \quad \begin{pmatrix} \cos(4t)\, t \\ \sin(4t)\, t \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t}, \quad \begin{pmatrix} -\sin(4t)\, t \\ \cos(4t)\, t \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}$$

□

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