Preparing Midterm #3

MATH 286 — Differential Equations Plus
April 17, 2014

The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka!" but "That's funny ...".
— Isaac Asimov (1920–1992) —

Problem 1. Let A be a 2×2 matrix such that $e^{At} = \begin{pmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (c+t)e^{rt} \end{pmatrix}$. What are the values of c and r?

Problem 2. Consider $\boldsymbol{x}' = A\boldsymbol{x}$ where $A = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$.

(a) Find the general solution.

(b) Find e^{At}.

(c) Find a particular solution to $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}$.

Problem 3. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at 15 gal/min. Mixed solution from T_1 is pumped into T_2 at 10 gal/min and from T_2 into T_3 at 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $x_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the x_i.

Problem 4. Let A be a 3×3 matrix such that $e^{At} = \begin{pmatrix} e^{2t} - te^{-t} & te^{-t} & -e^{2t} + (t+1)e^{-t} \\ e^{2t} - e^{-t} & e^{-t} & -e^{2t} + e^{-t} \\ -te^{-t} & te^{-t} & (t+1)e^{-t} \end{pmatrix}$.

(a) What are the eigenvalues of A? Indicate if an eigenvalue is repeated and what its defect is.

(b) Solve the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$.

(c) Find a particular solution to $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(d) Find A. Also, as a challenge, find A^{100}.

Problem 5. The 6×6 matrix A has eigenvalues $-3, -3, 0, 1, 1, 1$.

(a) Which eigenvalues can be defective? Briefly describe in *all* possible scenarios what sort of (generalized) eigenvectors would arise, and what form the solutions take in each case.

(b) We wish to solve $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 2t^2, & e^{-2t}\sin(t), & 0, & -1, & 0, & t\cos(t) \end{pmatrix}^T$. Write down a particular solution $\boldsymbol{x}_p$ with undetermined coefficients. It should have as few terms as possible and still work for any matrix A with the stated eigenvalues.

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Problem 6. Consider $\boldsymbol{x}' = A\boldsymbol{x}$ where $A = \begin{pmatrix} 2 & 4 & -1 \\ 7 & -1 & -5 \\ -1 & 1 & -1 \end{pmatrix}$.

(a) Find a fundamental matrix. *Hint:* The eigenvalues of A are $-3, -3, 6$.

(b) Solve the initial value problem with $\boldsymbol{x}(0) = (\ 3 \ \ 0 \ \ 0\ )^T$.

Problem 7. Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Show that the matrix $N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is nilpotent.

(b) Use the fact that N is nilpotent, to find e^{At}.

(c) Solve the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$, $\quad \boldsymbol{x}(0) = (\ 1 \ \ 2 \ \ 3\ )^T$.

(d) Find a particular solution of $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix}$.

(e) Use a different method to solve the previous problem.

Problem 8. Find the general solution of

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \boldsymbol{x}.$$

You may use that the characteristic polynomial has the repeated roots $1 \pm i$. The general solution should be given in terms of real-valued functions.

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