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MATH 286 SECTION X1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 3

November 20, 2013

INSTRUCTOR: M. BRANNAN

INSTRUCTIONS

- This exam 60 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- GOOD LUCK!

PLEASE NOTE: "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."

USEFUL FORMULAS:

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \dots$$

$$\mathbf{x}(t) = \Phi(t)\Phi(a)^{-1}\mathbf{x}(a) + \Phi(t) \int_a^t \Phi(s)^{-1}\mathbf{f}(s)ds$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Question:	1	2	3	Total
Points:	12	14	24	50
Score:				

1. Consider the following first order linear system of differential equations:

$$\begin{aligned} x_1' &= -3x_1 + 2x_3 \\ x_2' &= x_1 - x_2 \\ x_3' &= -2x_1 - x_2. \end{aligned}$$

(a) (4 points) Write this system in the vector-matrix form $\mathbf{x}' = A\mathbf{x}$.

Solution:

$$\mathbf{x}' = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) (8 points) The eigenvalues of the matrix A in part (a) are -2 and $-1 \pm (\sqrt{2})i$. An eigenvector associated to the eigenvalue $-1-(\sqrt{2})i$ is

$$\mathbf{w} = \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i. \end{bmatrix}.$$

Find three linearly independent *real-valued* solutions to this system.

Solution: Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an eigenvector for $\lambda_1 = -2$. Then

$$(A+2I)\mathbf{v}_1 = 0 \iff \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

The first two equations imply that $a = 2c = -b$, while the third equation is the first minus the second. Taking $a = 1$, we get an eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1/2 \end{bmatrix}$. This gives one solution $\mathbf{x}_1(t) = e^{-2t}\mathbf{v}$.

We are also given the eigenvector $\mathbf{w}$ associated to the eigenvalue $-1-(\sqrt{2})i$.

This yields a complex solution

$$\mathbf{z}(t) = e^{(-1-\sqrt{2}i)t}\mathbf{w} = e^{-t}(\cos\sqrt{2}t - i\sin\sqrt{2}t)\begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i. \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} -\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t. \end{bmatrix}}_{\mathbf{x}_2(t)} + i\,\underbrace{e^{-t}\begin{bmatrix} -\sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t. \end{bmatrix}}_{\mathbf{x}_3(t)}$$

Then $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$ are three linearly independent solutions real valued solutions.

2. (a) (10 points) Let λ be a fixed real number, and let

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Show that $e^{tA} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$.

Solution: Write $tA = \lambda tI + tN$, where

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies N^3 = 0.$$

From this, we get

$$e^{tA} = e^{\lambda tI + tN} = e^{\lambda tI}e^{tN} = e^{\lambda t}Ie^{tN} = e^{\lambda t}(I + tN + \frac{t^2}{2}N^2) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

An alternate solution to this problem would be to show that the matrix $\Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$ is a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$, and that $\Phi(0) = I$. Then $e^{tA} = \Phi(t)\Phi(0)^{-1} = \Phi(t)$.

(b) (4 points) Let A be the matrix from part (a). Solve the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t); \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The solution is

$$\mathbf{x}(t) = e^{tA}\mathbf{x}(0) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 + 2t + \frac{3t^2}{2} \\ 2 + 3t \\ 3 \end{bmatrix}.$$

3. (a) (9 points) Find two linearly independent solutions to the system

$$\mathbf{x}'(t) = A\mathbf{x}(t); \quad \text{where} \quad A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

Solution: First, we find eigenvalues:

$$0 = \det(A - \lambda I) = (7-\lambda)(3-\lambda) + 4 = 21 - 10\lambda + \lambda^2 + 4 = \lambda^2 - 10\lambda + 25.$$

So $\lambda = 5$ is a multiplicity 2 eigenvalue. Next, we look for eigenvectors: Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be an eigenvector, then

$$(A - 5I)\mathbf{v} = 0 \iff \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \mathbf{v} = \begin{bmatrix} a \\ -2a \end{bmatrix} \qquad (a \neq 0).$$

From this we see that $\lambda = 5$ has defect 1 and we want to find a length 2 chain $\{\mathbf{v}_1, \mathbf{w}_2\}$ of generalized eigenvectors. Taking $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, we then have for $\mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$,

$$\mathbf{v}_1 = (A - 5I)\mathbf{v}_2 \iff \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \iff \begin{matrix} 1 = 2a + b \\ -2 = -4a - 2b \end{matrix}.$$

Taking $a = 1/2$ and $b = 0$, we get $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$. This gives the following two linearly independent solutions:

$$\mathbf{x}_1(t) = e^{5t}\mathbf{v}_1 \quad \& \quad \mathbf{x}_2(t) = e^{5t}(t\mathbf{v}_1 + \mathbf{v}_2).$$

(b) (3 points) Write down a fundamental matrix $\Phi(t)$ for the system in part (a).

Solution: Let $\Phi(t) = [\mathbf{x}_1(t)\ \mathbf{x}_2(t)] = \begin{bmatrix} e^{5t} & e^{5t}(t + 1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix}$.

(c) (5 points) Compute the matrix exponential e^{tA}, where A is the matrix from part (a).

Solution:

$$\begin{aligned} e^{tA} = \Phi(t)\Phi(0)^{-1} &= \begin{bmatrix} e^{5t} & e^{5t}(t+1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ -2 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{5t} & e^{5t}(t+1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \end{aligned}$$

(d) (7 points) Solve the following initial value problem:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}; \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where A is the matrix from part (a).

Solution: We will use the matrix exponential from part (c).

$$\begin{aligned} \mathbf{x}(t) &= e^{tA}\mathbf{x}(0) + e^{tA}\int_0^t e^{-sA}\begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds \\ &= e^{tA}(\mathbf{x}(0) + \int_0^t e^{-sA}\begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-5s}(-2s+1) & -se^{-5s} \\ e^{-5s}(4s) & e^{-5s}(2s+1) \end{bmatrix} \begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds \right) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-6s}(-2s+1) \\ e^{-6s}(-4s) \end{bmatrix} ds \right) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{9}e^{-6t}(3t-1)+\frac{1}{9} \\ \frac{-1}{9}e^{-6t}(6t+1)+\frac{1}{9} \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{5t}(3t+1) \\ e^{5t}(-6t+1) \end{bmatrix} + \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \begin{bmatrix} \frac{1}{9}e^{-6t}(3t-1)+\frac{1}{9} \\ \frac{-1}{9}e^{-6t}(6t+1)+\frac{1}{9} \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{9}e^{5t}(3t+1) \\ \frac{10}{9}e^{5t}(-6t+1) \end{bmatrix} + \frac{e^{-t}}{9}\begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{aligned}$$

(**BONUS PROBLEM (5 Points)**).

Let λ be an eigenvalue of an $n \times n$ matrix A and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a length r chain of generalized eigenvectors associated to the eigenvalue λ.

(a). Explain what it means to be a length r chain of generalized eigenvectors.

For $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ to be a length r chain of generalized eigenvectors, the vector $\mathbf{v}_r$ must satisfy $(A-\lambda I)^r \mathbf{v}_r = 0$, $(A-\lambda I)^{r-1}\mathbf{v}_r \neq 0$, and the remaining vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$ are then given by

$$\mathbf{v}_s = (A-\lambda I)^{r-s}\mathbf{v}_r \neq 0 \qquad (1 \le s \le r-1).$$

(b). Show that the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ are linearly independent. (**Hint**: Suppose that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r = 0$. Multiply this equation by $(A-\lambda I), (A-\lambda I)^2, (A-\lambda I)^3$, etc... and see what happens.)

Suppose that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r = 0$. Note that

$$(A-\lambda I)^k \mathbf{v}_s = 0 \qquad (k \ge s),$$

and

$$(A-\lambda I)^k \mathbf{v}_s = \mathbf{v}_{s-k} \qquad (s > k).$$

Therefore if we take the above equation and multiply it by $(A-\lambda I)^k$ for $k = 1, 2, \ldots, r-1$, we get the following system of equations

$$\begin{aligned}
&0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r \\
&0 = c_2\mathbf{v}_1 + \ldots + c_r\mathbf{v}_{r-1} \quad (\textit{mult. by } A-\lambda I) \\
&0 = c_3\mathbf{v}_1 + \ldots + c_r\mathbf{v}_{r-2} \quad (\textit{mult. by } (A-\lambda I)^2) \\
&\ldots \\
&0 = c_{r-1}\mathbf{v}_1 + c_r\mathbf{v}_2 \quad (\textit{mult. by } (A-\lambda I)^{r-2}) \\
\Longrightarrow\ &0 = c_r\mathbf{v}_1 \quad (\textit{mult. by } (A-\lambda I)^{r-1})
\end{aligned}$$

The last equation implies that $c_r = 0$, the second last then implies that $c_{r-1} = 0$, and continuing up the list of equations, we see that $c_1 = c_2 = \ldots = c_r = 0$. Therefore the given vectors are linearly independent.

(*Extra work space.*)