Boundary value problems and partial differential equations

Endpoint problems and eigenvalues

Example 181. The IVP (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$. ♢

Initial value problems are often used when the problem depends on time. Then, as we know for instance for the motion of a pendulum, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next two examples illustrate that such boundary value problem may or may not have unique solutions.

Example 182. The BVP (boundary value problem) $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, already using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well. ♢

Example 183. The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition true for any B. ♢

It is therefore natural to ask: for which λ does $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ have nonzero solutions? Such solutions are called eigenfunctions and λ is the corresponding eigenvalue.

Remark 184. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue. ♢

Example 185. Find all eigenfunctions and eigenvalues of $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.

Such a problem is called an eigenvalue problem.

Solution. The solutions of the DE look different in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, so we consider them individually.

$\boldsymbol{\lambda = 0}$. Then $y(x) = Ax + B$ and $y(0) = y(L) = 0$ implies that $y(x) = 0$. No eigenfunction here.

$\boldsymbol{\lambda < 0}$. Write $\lambda = -\rho^2$. Then $y(x) = Ae^{\rho x} + Be^{-\rho x}$. $y(0) = A + B \overset{!}{=} 0$ implies $B = -A$. Using that, we get $y(L) = A(e^{\rho L} - e^{-\rho L}) \overset{!}{=} 0$. For eigenfunctions we need $A \neq 0$, so $e^{\rho L} = e^{-\rho L}$ which implies $\rho L = -\rho L$. This cannot happen since $\rho \neq 0$ and $L \neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda > 0}$. Write $\lambda = \rho^2$. Then $y(x) = A\cos(\rho x) + B\sin(\rho x)$. $y(0) = A \overset{!}{=} 0$. Using that, $y(L) = B\sin(\rho L) \overset{!}{=} 0$. Since $B \neq 0$ for eigenfunctions, we need $\sin(\rho L)$. This happens if $\rho L = n\pi$ for $n = 0, 1, 2, \ldots$. Consequently, we do find the eigenfunctions $y_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \ldots$, with eigenvalue $\lambda = (\frac{n\pi}{L})^2$. ♢

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